(*) The definite integral

Recall that the ***definite integral*** of the function $y = f(x)$ on the interval $[a, b]$ is denoted by

$$\int_a^b f(x)\,dx$$

and is defined by the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_j^*)\cdot \Delta x_j\right),$$

assuming that the limit exists. In this definition we assume that for each n:

(i) $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
The collection of subintervals $\{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$ is called a ***partition*** of the interval $[a, b]$.

(ii) x_j^* is *a point* (chosen as we please) in the interval $[x_{j-1}, x_j]$, i.e., $x_{j-1} \le x_j^* \le x_j$.

(iii) $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$. This is the *length* of the j^{th} subinterval, $[x_{j-1}, x_j]$.

(iv) $\lim_{n\to\infty} \delta_n = 0$, where

$$\delta_n = \max_{1\le j\le n} \Delta_j$$

(i.e., all of the subintervals grow thinner and thinner as $n \to \infty$).

Problem: Evaluating the sums that appear in the definition of the definite integral can be difficult (if not impossible).

Solution: ....

(*) The Fundamental Theorem of Calculus (FTC)

It turns out that there is a deep connection between definite integrals (which are limits of sums) and indefinite integrals (which are collections of antiderivatives). Specifically, if

$$\int f(x)\,dx = F(x) + C$$

(i.e., if $F'(x) = f(x)$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

There are several ways to see why this is so. The book (in section 14.7) illustrates one of the arguments,† and in class we used the following argument.

†If $f(x)$ is continuous, and we define $F(x) = \int_a^x f(t)\,dt$, then it follows (as demonstrated in section 14.7) that $F'(x) = f(x)$ and from this it follows that $\int_a^b f(x)\,dx = F(b) - F(a)$ (as demonstrated in section 14.7).

(i) If we divide the interval $[a, b]$ into n subintervals with endpoints

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

(as in the definition of the definite integral), then

$$\begin{aligned} F(b) - F(a) &= \sum_{j=1}^{n} F(x_j) - F(x_{j-1}) \\ &= (\cancel{F(x_1)} - F(x_0)) + (\cancel{F(x_2)} - \cancel{F(x_1)}) + (\cancel{F(x_3)} - \cancel{F(x_2)}) + \cdots + (F(x_n) - \cancel{F(x_{n-1})}), \end{aligned}$$

because all of the terms $F(x_1), F(x_2), \ldots, F(x_{n-1}$ in the sum on the right cancel except the $-F(x_0) = -F(a)$ in the first term and the $+F(x_n) = +F(b)$ in the last term.

(ii) If $\Delta x_j = x_j - x_{j-1}$ is small, then $F(x_j) - F(x_{j-1} \approx F'(x_{j-1})\Delta x_j$, as follows from *linear approximation*. Therefore, if n is very large and all of the Δx_js are very small, then

$$\begin{aligned} F(b) - F(a) &= \sum_{j=1}^{n} F(x_j) - F(x_{j-1}) \\ &\approx \sum_{j=1}^{n} F'(x_{j-1})\Delta x_j = \sum_{j=1}^{n} f(x_{j-1})\Delta x_j, \end{aligned}$$

because $F'(x) = f(x)$ by assumption.

(iii) Cutting out the middle man, we see that if n is large (and the Δx_js are all small), then

$$F(b) - F(a) \approx \sum_{j=1}^{n} f(x_{j-1})\Delta x_j$$

This approximation becomes more and more accurate as $n \to \infty$, and since $F(b) - F(a)$ is *constant*, it follows that

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_{j-1})\Delta x_j = F(b) - F(a).$$

By definition however,

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_{j-1})\Delta x_j = \int_a^b f(x)\, dx$$

(another constant), and it must be therefore that

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

Which is what the FTC says.

(*) **Examples**

(a) Observe that $\int x^2\,dx = \dfrac{x^3}{3} + C$, so it follows from the FTC that

$$\int_1^3 x^2\,dx = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}.$$

Notation: We use the following notation

$$F(x)\Big|_a^b = F(b) - F(a)$$

to express the difference $F(b) - F(a)$. It makes applying the FTC less cumbersome, in that we don't need to write down an indefinite integral separately before calculating the definite integral we want. E.g., in the example above, we write

$$\int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}.$$

(b) Calculate the definite integral $\displaystyle\int_1^4 \frac{\sqrt{x}-2}{\sqrt{x}}dx$

$$\int_1^4 \frac{x-2}{\sqrt{x}}dx = \int_1^4 x^{1/2} - 2x^{-1/2}\,dx = \left(\frac{2}{3}x^{3/2} - 4x^{1/2}\right)\Big|_1^4 = \overbrace{\left(\frac{16}{3} - 8\right)}^{\text{evaluate at 4}} - \overbrace{\left(\frac{2}{3} - 4\right)}^{\text{evaluate at 1}} = \frac{2}{3}.$$

(c) A firm's marginal revenue function is given by $\dfrac{dr}{dq} = \sqrt{400 - 0.1q}$. Find the change in the firm's revenue when output increases from $q_1 = 1750$ to $q_2 = 3000$.

We want to find $r(q_2) - r(q_1)$, where $r(q)$ is the firm's revenue function, and according to the FTC

$$r(q_2) - r(q_1) = \int_{q_1}^{q_2} \frac{dr}{dq}\,dq.$$

Now,

$$\int \frac{dr}{dq}\,dq = \int \sqrt{400 - 0.1q}\,dq = -10\int u^{1/2}\,du = -\frac{20}{3}u^{3/2} + C = -\frac{20}{3}(400 - 0.1q)^{3/2} + C,$$

using the substitution $u = (400 - 0.1q)$, and $du = -0.1dq$, so $dq = -10du$. This means that

$$\begin{aligned} r(3000) - r(1750) &= \int_{1750}^{3000} \sqrt{400 - 0.1q}\,dq \\ &= -\frac{20}{3}(400 - 0.1q)^{3/2}\Big|_{1750}^{3000} \\ &= -\frac{20}{3}\left(100^{3/2} - 225^{3/2}\right) \approx 15833.33 \end{aligned}$$