

(*) **The definite integral: basic properties**

The most important ‘property’ of definite integrals is the *Fundamental Theorem of Calculus*, namely

(0) If $\int f(x) dx = F(x) + C$, (i.e., if $F'(x) = f(x)$), then

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a).$$

The next two properties can be seen as carry-overs from the properties of indefinite integrals (given the FTC) on the one hand, or from the properties of sums (given the *definition* of the definite integral) on the other. They can be justified using the definition or using the FTC.

$$(1) \int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$$

$$(2) \int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx.$$

The last two properties are specific to definite integrals — they don’t have indefinite integral counterparts (though they do have summation counterparts). These two properties may also be justified using either the FTC or the definition of the definite integral.

$$(3) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

I.e., the *interval of integration* $[a, b]$ may be broken into subintervals $[a, c]$ and $[c, b]$.

$$(4) \int_b^a f(x) dx = - \int_a^b f(x) dx.$$

This last property highlights the fact that a definite integral depends on (i) the integrand $f(x)$, (ii) the interval $[a, b]$ and (iii) the *orientation*, i.e., the *direction* in which we are integrating — from a to b or from b to a . Perhaps more importantly, this property is a reminder that the upper limit of integration is not required to be larger than the lower limit of integration.

Example. $\int_0^3 x^2 dx = \frac{x^3}{3} \Big|_0^3 = \frac{27}{3} - \frac{0}{3} = 9$ and $\int_3^0 x^2 dx = \frac{x^3}{3} \Big|_3^0 = \frac{0}{3} - \frac{27}{3} = -9.$

(*) **Substitution in a definite integral**

When we use the substitution $u = g(x)$ to simplify the indefinite integral $\int f(g(x))g'(x) dx$, every x -element of the original integral needs to be replaced by a corresponding u -element in the new integral, i.e.,

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

Substitution in a *definite* integral follows the same principle, which means that the *limits of integration* need to change as well because they are the limits for the variable x . In the integral

$$\int_a^b f(g(x))g'(x) dx,$$

the variable x goes from a to b . If we make the substitution $u = g(x)$, $du = g'(x) dx$, then as the variable x goes from a to b , the variable $u = g(x)$ goes from $g(a)$ to $g(b)$, so that

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Example. Compute the integral $\int_0^{20} 400e^{-0.1x} dx$.

To do this, we use the substitution $u = -0.1x$, which entails $du = -0.1 dx$ so $dx = -10 du$, also, when $x = 0$, $u = -0.1 \cdot 0 = 0$ and when $x = 20$, $u = -0.1 \cdot 20 = -2$. Therefore,

$$\begin{aligned} \int_0^{20} 400e^{-0.1x} dx &= -10 \int_0^{-2} 400e^u du \\ &= -4000e^u \Big|_0^{-2} = -4000(e^{-2} - e^0) \\ &= 4000(1 - e^{-2}) \approx 3458.659. \end{aligned}$$

(*) **Application: the area between two curves**

Consider the region R , bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, as depicted in the Figure 1 below.

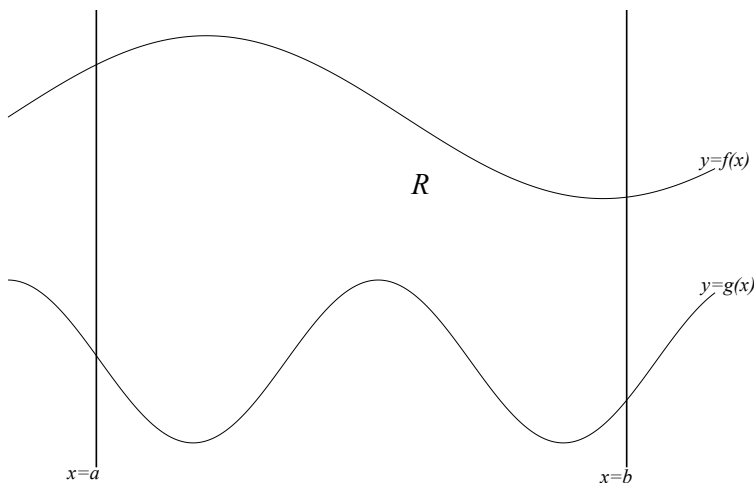


Figure 1: The region R

To calculate the area of R , we follow the same procedure we used in the special case when the lower boundary of the region was the curve $y = 0$ (the x -axis). We divide the interval $[a, b]$ into subintervals, $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ (where $a = x_0$ and $b = x_n$), and approximately cover the region R with rectangles, r_1, r_2, \dots, r_n , as depicted in Figure 2.

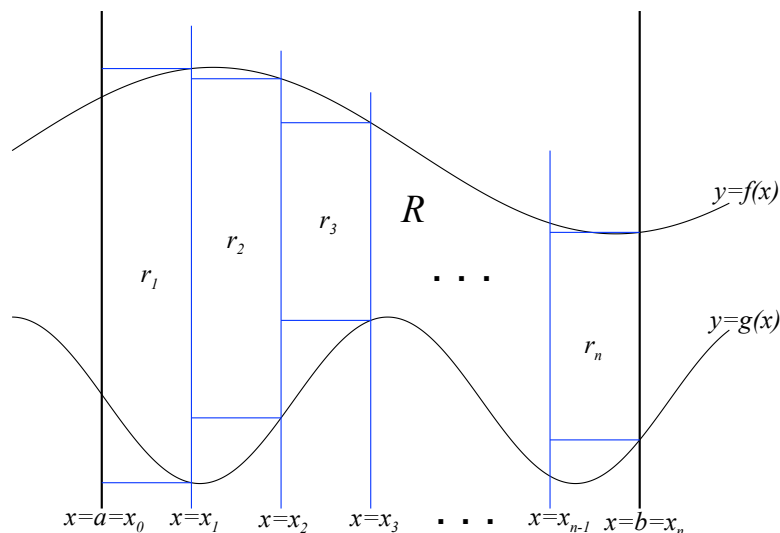


Figure 2: The region R , approximately covered by rectangles.

As before, we see that the area of R is approximately equal to the sum of the areas of the rectangles r_1, r_2, \dots, r_n , i.e.,

$$\text{area}(R) \approx \sum_{j=1}^n \text{area}(r_j).$$

In the figure above, we used the right-hand endpoints of the intervals to determine the heights of the rectangles so that $\text{height}(r_j) = (f(x_j) - g(x_j))$, and since the width of r_j is $\Delta x_j = x_j - x_{j-1}$ (as usual), it follows that $\text{area}(r_j) = (f(x_j) - g(x_j))\Delta x_j$, so

$$\text{area}(R) \approx \sum_{j=1}^n (f(x_j) - g(x_j))\Delta x_j.$$

Also as before, we see that as the number n of rectangles grows larger, while the widths of rectangles simultaneously shrink, the approximation above to the area of R becomes more and more accurate, and we conclude that

$$\text{area}(R) = \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n (f(x_j) - g(x_j))\Delta x_j \right).$$

Finally, we recognize the limit on the left as a definite integral and we can summarize this discussion by saying:

If $f(x) \geq g(x)$ throughout the interval $[a, b]$, then the area of the region R bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is given by

$$\text{area}(R) = \int_a^b f(x) - g(x) dx.$$