(*) **Population growth models — Exponential growth**

The simplest model for population growth is based on the assumption that the population grows *at a rate proportional to its size.* This assumption considers only the factors intrinsic to the population itself, e.g., birthrate, and leads to the differential equation

$$\frac{dP}{dt} = rP,$$

where $P(t)$ is the size of the population at time t, and r is the *intrinsic growth rate.* This equation is easy to solve (after separating the variables):

$$\frac{dP}{P} = r\,dt \implies \int \frac{dP}{P} = r\int dt \implies \ln P = rt + C \implies P = Ae^{rt},$$

where

- $P > 0$, so we can drop the absolute value sign.
- $A = e^C$, and in fact...
- $A = P(0) = P_0$, the initial population size.

I.e., the *exponential growth* model is

$$P(t) = P_0 e^{rt}.$$

Example 5. The population of a small island in the year 1950 was 870 people, and in the year 2000, the population was 1250. Assuming exponential growth, what will the island's population be in the year 2050? How about in 2150?

Based on the assumption of exponential growth, we have

$$P(t) = 870e^{rt},$$

with time being measured in years, and $t = 0$ corresponding to the year 1950. This means that

$$\begin{aligned} 1250 = P(50) = 870e^{50r} &\implies e^{50r} = \frac{1250}{870} \\ &\implies 50r = \ln(125/87) \\ &\implies r = \frac{1}{50}\ln(125/87) \quad (\approx 0.00725) \end{aligned}$$

Therefore

$$P(100) = 870e^{100r} \approx 1796 \quad \text{and} \quad P(200) = 870e^{200r} \approx 3708.$$

The exponential growth model $P = P_0e^{rt}$ can be quite accurate in the short run, but not in the long run, because an exponentially growing population will eventually outstrip its resources. This observation leads to a different model.

(*) **Population growth models — Logistic growth**

This model accounts for the fact that populations grow in environments that have limited resources. Such an environment has a ***carrying capacity***, which is the maximum (sustainable) size for the population growing there.

The logistic model is based on the following assumptions/requirements.

(i) When the population is small *relative to the carrying capacity*, it should grow at a rate (approximately) proportional to its size (like exponential growth).

(ii) As the population gets close to the carrying capacity in size, the growth rate should approach 0.

(iii) If the initial population size is bigger than the carrying capacity, the growth rate should be negative.

(iv) The model should be as simple as possible.

If the carrying capacity is M and the intrinsic growth rate is r, then the first three assumptions translate to

(i) If $P/M \approx 0$, then $\dfrac{dP}{dt} \approx rP$.

(ii) If $P/M \approx 1$, then $\dfrac{dP}{dt} \approx 0$.

(iii) If $P/M > 1$, then $\dfrac{dP}{dt} < 0$.

These assumptions (and the desire for as simple a model as possible), lead to the *logistic equation*:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right),$$

which satisfies all three conditions:

(*) If $P/M \approx 0$, then $rP\left(1 - \frac{P}{M}\right) \approx rP(1-0) = rP$

(*) If $P/M \approx 1$, then $rP\left(1 - \frac{P}{M}\right) \approx rP(1-1) = 0$

(*) If $P/M > 1$, then $rP\left(1 - \frac{P}{M}\right) < 0$

The logistic equation is separable and is solved as follows.

First, factor out $1/M$ from the second factor on the right

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) = \frac{r}{M}P\left(M - P\right).$$

Then separate

$$\frac{dP}{P(M-P)} = \frac{r}{M}\,dt.$$

Then integrate (using formula #5 in the appendix, with $a = M$ and $b = -1$)

$$\int \frac{dP}{P(M-P)} = \int \frac{r}{M}\,dt \implies \frac{1}{M}\ln\left|\frac{P}{M-P}\right| = \frac{rt}{M} + C.$$

Finally, solve for P

$$\frac{1}{\cancel{M}} \ln\left|\frac{P}{M-P}\right| = \frac{rt}{\cancel{M}} + C \implies \ln\left|\frac{P}{M-P}\right| = rt + C$$
$$\implies \frac{P}{M-P} = Ae^{rt}$$

where $A = \pm e^C$.

A little more algebra:

$$P = (M-P)Ae^{rt} = AMe^{rt} - APe^{rt} \implies P + APe^{rt} = AMe^{rt}$$
$$\implies P(1 + Ae^{rt}) = AMe^{rt}$$
$$\implies P = \frac{AMe^{rt}}{1 + Ae^{rt}}$$

The formula for $P(t)$ can be further manipulated in different ways.

One approach is to divide the numerator and denominator by Ae^{rt} which gives

$$P = \frac{M}{1 + be^{-rt}},$$

where $b = A^{-1}$. (Our textbook does it this way.)

Another approach is to replace A by a more meaningful parameter. Both M and r have meaningful interpretations, and it is relatively easy to express A in terms of M and the *initial population size* P_0.

If $t = 0$, then

$$P_0 = P(0) = \frac{AM}{1+A} \implies AM = P_0(1+A) = P_0 + AP_0$$
$$\implies AM - AP_0 = P_0 \implies A(M - P_0) = P_0$$
$$\implies A = \frac{P_0}{M - P_0}$$

Now, substitute this for A in the first expression for P

$$P = \frac{AMe^{rt}}{1 + Ae^{rt}} \implies \frac{\frac{P_0}{M-P_0} Me^{rt}}{1 + \frac{P_0}{M-P_0} e^{rt}}$$

Finally, multiply both top and bottom by $(M - P_0)e^{-rt}$, which gives

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}}.$$

Example. A new virus is spreading on a closed network of 5000 computers. By the time the virus is first spotted, 25 computers are infected, and two hours later 200 computers are infected. Assuming logistic growth, how many hours before half the network is infected?

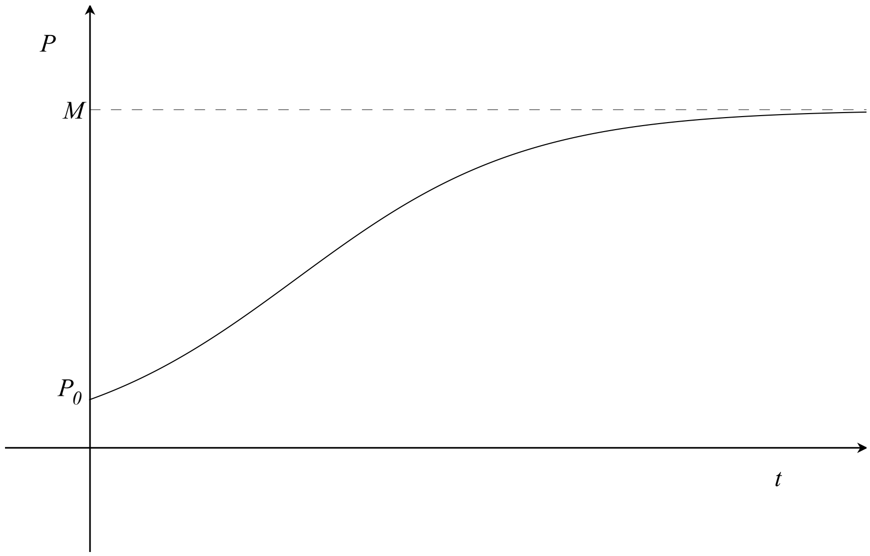

Figure 1: Graph of $P = \dfrac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$

In this example, we know the carrying capacity $M = 5000$ and the initial population size $P_0 = 25$, so the number of infected computers at time t is

$$P(t) = \frac{25 \cdot 5000}{25 + 4975e^{-rt}} = \frac{5000}{1 + 199e^{-rt}}.$$

From the data, we have

$$\begin{aligned} P(2) = \frac{5000}{1 + 199e^{-2r}} = 200 &\implies 5000 = 200(1 + 199e^{-2r}) \\ &\implies 25 = 1 + 199e^{-2r} \\ &\implies 24 = 199e^{-2r} \\ &\implies e^{2r} = \frac{199}{24} \\ &\implies r = \frac{1}{2}\ln(199/24) \end{aligned}$$

Finally, solve the equation $P(t_1) = 5000/2 = 2500$:

$$2500 = \frac{5000}{1 + 199e^{-rt_1}} \implies 1 + 199e^{-rt_1} = \frac{5000}{2500} = 2$$

$$\implies 199e^{-rt_1} = 1 \implies e^{rt_1} = 199$$

$$\implies t_1 = \frac{\ln 199}{r} = \frac{\ln 199}{\frac{1}{2}\ln(199/24)} \approx 5$$

Conclusion: Half the network will be infected about 5 hours after the virus is first detected.