(*) Linear approximation, more generally

Our story thus far... We have already learned that given a function $z = f(x, y)$ and a point (x_0, y_0), if $\Delta x \approx 0$ and $\Delta z = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$ (so y is held fixed), then

$$\Delta z \approx \left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta x.$$

Likewise, if $\Delta z = f(x_0, y_0 + \Delta y) - f(x_0, y_0)$ and $\Delta y \approx 0$ (and x is not changing), then

$$\Delta z \approx \left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta y.$$

These two approximation formulas naturally lead to the question: *what if both variables, x and y, change a little? I.e., what can we say about the change*

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0),$$

if both $\Delta x \approx 0$ and $\Delta y \approx 0$ (but neither is equal to 0)?

The answer is a simple generalization of the approximation formulas above. Namely, if $\Delta x \approx 0$ and $\Delta y \approx 0$, then

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \approx \left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta x + \left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta y. \tag{1}$$

This fact generalizes to any number of variables. For example, if $w = g(x, y, z)$ and

$$\Delta w = g(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - g(x_0, y_0, z_0),$$

then

$$\Delta w \approx \left.\frac{\partial w}{\partial x}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} \cdot \Delta x + \left.\frac{\partial w}{\partial y}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} \cdot \Delta y + \left.\frac{\partial w}{\partial z}\right|_{\substack{x=x_0\\ y=y_0\\ z=z_0}} \cdot \Delta z \tag{2}$$

as long as $\Delta x \approx 0$, $\Delta y \approx 0$ and $\Delta z \approx 0$.

Example. A firm produces two related goods A and B. The variables q_A and q_B are the quantities of these products that the firm produces each month, both measured in 100s of units, and the firm's joint cost function for producing these goods is

$$C = 0.04q_A^2 + 0.06q_Aq_B + 0.05q_B^2 + 20q_A + 25q_B + 50,$$

where C is the monthly cost, measured in \$1000s.

The firm is currently producing 1000 units of A and 1000 units of B,[†] and the current monthly cost of production is

$$C\Big|_{\substack{q_A=10\\ q_B=10}} = 0.04 \cdot 10^2 + 0.06 \cdot 10^2 + 0.05 \cdot 10^2 + 20 \cdot 10 + 25 \cdot 10 + 50 = 110,$$

[†]So $q_A = q_B = 10$.

i.e., the firm's monthly cost is \$110,000.

The current *marginal costs of products A and B* are

$$\left.\frac{\partial C}{\partial q_A}\right|_{\substack{q_A=10\\q_B=10}} = \left.0.08q_A + 0.06q_B + 20\right|_{\substack{q_A=10\\q_B=10}} = 0.8 + 0.6 + 20 = 21.4$$

and

$$\left.\frac{\partial C}{\partial q_B}\right|_{\substack{q_A=10\\q_B=10}} = \left.0.06q_A + 0.1q_B + 25\right|_{\substack{q_A=10\\q_B=10}} = 0.6 + 1 + 25 = 26.6$$

respectively.

If the firm increases the production of good A by 10 units a month and increases the production of good B by 20 units a month, then

$$\Delta q_A = \frac{10}{100} = 0.1 \ \text{ and } \ \Delta q_B = \frac{20}{100} = 0.2$$

and

$$\Delta C \approx \left.\frac{\partial C}{\partial q_A}\right|_{\substack{q_A=10\\q_B=10}} \Delta q_A + \left.\frac{\partial C}{\partial q_B}\right|_{\substack{q_A=10\\q_B=10}} \Delta q_B = 21.4 \cdot 0.1 + 26.6 \cdot 0.2 = 7.46,$$

as follows from the (more) general linear approximation formula (1). That is, the firm's monthly cost will increase by about \$7460.00.

(*) **The linear Taylor polynomial in several variables.**

We found that a function of one variable, $y = f(x)$, can be approximated by a linear function $T_1(x)$ in the vicinity of any point where $f(x)$ is differentiable. Specifically, we found that if f is differentiable at x_0 and $x \approx x_0$, then

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = T_1(x). \tag{3}$$

The linear function $T_1(x)$ is called the linear Taylor polynomial for $f(x)$ centered at x_0 and the graph $y = T_1(x)$ is the *tangent line* to the curve $y = f(x)$ at the point $(x_0, f(x_0))$.

The approximation in Equation (3), is just another way of viewing the approximation

$$\Delta y \approx \left.\frac{dy}{dx}\right|_{x=x_0} \cdot \Delta x,$$

where $\Delta y = f(x) - f(x_0)$ and $\Delta x = x - x_0$, and we can rewrite the linear approximation formulas (1) and (2) in the same way, as follows below.

All we have to do is rename and rearrange. First consider the linear approximation formula (1) for the function $z = f(x, y)$. If we write $\Delta x = x - x_0$ and $\Delta y = y - y_0$, then $x_0 + \Delta x = x$ and $y_0 + \Delta y = y$, then the approximation (1) can be written as

$$f(x, y) - f(x_0, y_0) \approx f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0),$$

(replacing the $\partial z/\partial x$ notation with the f_x notation for convenience). Now, adding $f(x_0, y_0)$ to both sides of the approximation above gives

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0). \tag{4}$$

The linear function $T_1(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$ is called the *linear Taylor polynomial* for $f(x,y)$ centered at (x_0,y_0).

This general form of linear approximation applies to any number of variables. E.g., returning to the three-variable function, $w = g(x,y,z)$ and if we write $x_0 + \Delta x = x$, $y_0 + \Delta y = y$ and $z_0 + \Delta z = z$, so that

$$\Delta x = x - x_0,\ \Delta y = y - y_0 \text{ and } \Delta z = z - z_0,$$

then the approximation formula (2) can be written

$$g(x,y,z) - g(x_0,y_0,z_0) \approx g_x(x_0,y_0,z_0)(x-x_0) + g_y(x_0,y_0,z_0)(y-y_0) + g_z(x_0,y_0,z_0)(z-z_0)$$

or

$$g(x,y,z) \approx g(x_0,y_0,z_0) + g_x(x_0,y_0,z_0)(x-x_0) + g_y(x_0,y_0,z_0)(y-y_0) + g_z(x_0,y_0,z_0)(z-z_0).$$

The linear function

$$T_1(x,y,z) = g(x_0,y_0,z_0) + g_x(x_0,y_0,z_0)(x-x_0) + g_y(x_0,y_0,z_0)(y-y_0) + g_z(x_0,y_0,z_0)(z-z_0)$$

is the ***linear Taylor polynomial*** for $w = g(x,y,z)$ centered at (x_0,y_0,z_0), and linear approximation (in three variables) can be written as

$$f(x,y,z) \approx T_1(x,y,z),$$

assuming that $x \approx x_0$, $y \approx y_0$ and $z \approx z_0$.

Example. Suppose that $g(x,y,z) = \sqrt[3]{x^2+2y^2+5z^2}$ and that $x_0 = y_0 = z_0 = 1$. Then, first of all, $g(1,1,1) = \sqrt[3]{1+2+5} = \sqrt[3]{8} = 2$. Next,

$$g_x = \frac{\partial}{\partial x}(x^2+2y^2+5z^2)^{1/3} = \frac{1}{3}(x^2+2y^2+5z^2)^{-2/3}\cdot 2x = \frac{2x}{3(x^2+2y^2+5z^2)^{2/3}},$$

and likewise,[‡]

$$g_y = \frac{4y}{3(x^2+2y^2+5z^2)^{2/3}} \text{ and } g_z = \frac{10z}{3(x^2+2y^2+5z^2)^{2/3}}.$$

Therefore,

$$g_x(1,1,1) = \frac{2\cdot 1}{3\cdot 8^{2/3}} = \frac{1}{6},\ g_y(1,1,1) = \frac{4\cdot 1}{3\cdot 8^{2/3}} = \frac{1}{3} \text{ and } g_z(1,1,1) = \frac{10\cdot 1}{3\cdot 8^{2/3}} = \frac{5}{6}.$$

It follows that the linear Taylor polynomial for $g(x,y,z)$ centered at $(1,1,1)$ is

$$T_1(x,y,z) = 2 + \frac{1}{6}(x-1) + \frac{1}{3}(y-1) + \frac{5}{6}(z-1).$$

[‡]Check!

For functions of one variable, we found that we could improve linear approximation by using quadratic approximation. Namely, if $f'(x)$ and $f''(x)$ are both defined at x_0 and

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2,$$

then

$$f(x) \approx T_2(x)$$

as long as $x \approx x_0$. To extend this idea to functions of several variables, we need to consider the second order partial derivatives of such functions.

(*) **Higher order partial derivatives**

The (*first order*) partial derivatives of a function $z = f(x, y)$ are

$$f_x = z_x = \frac{\partial z}{\partial x} \quad \text{and} \quad f_y = z_y = \frac{\partial z}{\partial y}.$$

The *second order* partial derivatives of a function $z = f(x, y)$ are (not surprisingly) the partial derivatives of its (first order) partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx} = (z_x)_x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = z_{yx} = (z_y)_x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = z_{xy} = (z_x)_y$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z_{yy} = (z_y)_y$$

Example. If $z = 4x^3 + 3x^2y - 2xy^2 + y^3$, then its (first order) partial derivatives are

$$z_x = 12x^2 + 6xy - 2y^2 \quad \text{and} \quad z_y = 3x^2 - 4xy + 3y^2$$

and its second order partial derivatives are

$$z_{xx} = 24x + 6y, \quad z_{yx} = 6x - 4y, \quad z_{xy} = 6x - 4y \quad \text{and} \quad z_{yy} = -4x + 6y.$$

Observation: In this example, $z_{xy} = z_{yx}$.

*Coincidence? ...**No.***

Fact:

> *Second and higher order partial derivatives do not depend on the order with respect to which a function is differentiated, only on **the number of times** the function is differentiated with respect to each variable.*

The *third order* partial derivatives of a function of two or more variables are the partial derivatives of its second order partial derivatives.

Notation: For $z = f(x, y)$

$$z_{xxx} = \frac{\partial^3 z}{\partial x^3}, \qquad \overbrace{z_{xyx} = z_{xxy}}^{\text{aforementioned fact}} = \frac{\partial^3 z}{\partial x^2 \partial y}, \quad \textbf{\textit{etc.}}$$

Example. (continued) For $z = 4x^3 + 3x^2y - 2xy^2 + y^3$ we already know that

$$z_{xx} = 24x + 6y, \quad z_{yx} = 6x - 4y = z_{xy} \quad \text{and} \quad z_{yy} = -4x + 6y.$$

so

$$z_{xxx} = 24, \; z_{yxx} = 6, \; z_{xyx} = 6, \; z_{yyx} = -4,$$
$$z_{xxy} = 6, \; z_{yxy} = -4, \; z_{xyy} = -4 \quad \text{and} \quad z_{yyy} = 6$$

Note that

$$z_{yxx} = z_{xyx} = z_{xxy} = 6 \quad \text{and} \quad z_{yyx} = z_{yxy} = z_{xyy} = -4,$$

as the 'Fact' predicted.