(*) **Quadratic Taylor polynomials in one variable.**

The *quadratic* Taylor polynomial $T_2(t)$ for $f(t)$, centered at t_0 is given by

$$T_2(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{f''(t_0)}{2}(t - t_0)^2,$$

which has the properties

- $T_1(t_0) = f(t_0)$,
- $T_1'(t_0) = f'(t_0)$ and
- $T_1''(t_0) = f''(t_0)$.

These key properties mean that $T_2(t)$ behaves very much like $f(t)$ at t_0, and as a consequence the approximation

$$f(t) \approx T_2(t)$$

is very accurate when t is close to t_0.

(*) **Quadratic Taylor polynomials in two variables**

Given a function $z = f(x, y)$ and a point (x_0, y_0), we we'll look for a *quadratic function in two variables* $T_2(x, y)$ that satisfies conditions analogous to those satisfied by $T_2(t)$ in the one-variable case.

Quadratic functions in two variables generally have the form

$$P(x, y) = \overbrace{\alpha}^{\text{constant}} + \overbrace{\beta x + \gamma y}^{\text{linear terms}} + \overbrace{\delta x^2 + \varepsilon xy + \phi y^2}^{\text{quadratic terms}}.$$

For the purposes of finding the quadratic Taylor polynomial $T_2(x, y)$ for the function $f(x, y)$, centered at the point (x_0, y_0), it is more convenient to write it as

$$T_2(x, y) = A + B(x - x_0) + C(y - y_0)$$
$$+ D(x - x_0)^2 + E(x - x_0)(y - y_0) + F(y - y_0)^2,$$

because it makes finding the values of the coefficients $(A, \ldots, F)$ *much* easier. The conditions that we want T_2 to satisfy are:

- $T_2(x_0, y_0) = f(x_0, y_0)$
- $T_{2x}(x_0, y_0) = f_x(x_0, y_0)$
- $T_{2y}(x_0, y_0) = f_y(x_0, y_0)$
- $T_{2xx}(x_0, y_0) = f_{xx}(x_0, y_0)$
- $T_{2xy}(x_0, y_0) = f_{xy}(x_0, y_0)$
- $T_{2yy}(x_0, y_0) = f_{yy}(x_0, y_0)$

These conditions lead to the following equations for the coefficients of $T_2(x, y)$...

Constant coefficient:

$$f(x_0, y_0) = T_2(x_0, y_0) = A + 0 + 0 + 0 + 0 + 0 = A \implies A = f(x_0, y_0).$$

Linear coefficients:

$$T_{2x} = B + 2D(x - x_0) + E(y - y_0)$$

and

$$T_{2y} = C + E(x - x_0) + 2F(y - y_0),$$

so

$$f_x(x_0, y_0) = T_{2x}(x_0, y_0) = B + 0 + 0 \implies B = f_x(x_0, y_0)$$

$$f_y(x_0, y_0) = T_{2y}(x_0, y_0) = C + 0 + 0 \implies C = f_y(x_0, y_0)$$

Quadratic terms:

$$T_{2xx} = 2D, \quad T_{2xy} = E \quad \text{and} \quad T_{2yy} = 2F,$$

so

$$f_{xx}(x_0, y_0) = T_{2xx}(x_0, y_0) = 2D \implies D = \frac{f_{xx}(x_0, y_0)}{2}$$

$$f_{xy}(x_0, y_0) = T_{2xy}(x_0, y_0) = E \implies E = f_{xy}(x_0, y_0)$$

$$f_{yy}(x_0, y_0) = T_{2yy}(x_0, y_0) = 2F \implies F = \frac{f_{yy}(x_0, y_0)}{2}$$

Conclusion:

The quadratic Taylor polynomial for $f(x, y)$, centered at (x_0, y_0) is

$$T_2(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$+ \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2$$

Quadratic approximation: If (x, y) is close to (x_0, y_0), then

$$f(x, y) \approx T_2(x, y).$$

Comment: '(x, y) is close to (x_0, y_0)' means that the distance in the plane between these two points is small. If both $|x - x_0|$ is small and $|y - y_0|$ is small, then the distance in the plane between (x, y) and (x_0, y_0),

$$\sqrt{(x - x_0)^2 + (y - y_0)^2},$$

will also be small.

(*) **Example**

Find the quadratic Taylor polynomial for

$$f(x, y) = (2x + 5y)^{1/2}$$

centered at the point $(x_0, y_0) = (3, 2)$.

This amounts to (a) finding the coefficients and (b) putting the pieces together correctly. To find the coefficients, we need to find the first and second order partial derivatives of $f(x, y)$:

$$f_x = (2x + 5y)^{-1/2}, \ f_y = \frac{5}{2}(2x + 5y)^{-1/2}, \ f_{xx} = -(2x + 5y)^{-3/2},$$

$$f_{xy} = -\frac{5}{2}(2x + 5y)^{-3/2} \quad \text{and} \quad f_{yy} = -\frac{25}{4}(2x + 5y)^{-3/2}.$$

Therefore

$$f(3, 2) = 16^{1/2} = 4, \quad f_x(3, 2) = 16^{-1/2} = \frac{1}{4}, \quad f_y(3, 2) = \frac{5}{2} \cdot 16^{-1/2} = \frac{5}{8}$$

$$f_{xx}(3, 2) = -16^{-3/2} = -\frac{1}{64}, \quad f_{xy}(3, 2) = -\frac{5}{2} \cdot 16^{-3/2} = -\frac{5}{128}$$

$$\text{and} \quad f_{yy}(3, 2) = -\frac{25}{4} \cdot 16^{-3/2} = -\frac{25}{256}.$$

So

$$T_2(x, y) = 4 + \frac{1}{4}(x - 3) + \frac{5}{8}(y - 2) - \frac{1}{128}(x - 3)^2 - \frac{5}{128}(x - 3)(y - 2) - \frac{25}{512}(y - 2)^2$$

We can use this to approximate

$$\sqrt{17} = f(3.25, 2.1) \dots$$

$$f(3.25, 2.1) = 4 + \frac{1}{4} \cdot \frac{1}{4} + \frac{5}{8} \cdot \frac{1}{10} - \frac{1}{128} \cdot \frac{1}{16} - \frac{5}{128} \cdot \frac{1}{4} \cdot \frac{1}{10} - \frac{25}{512} \cdot \frac{1}{100}$$

$$= 4 + \frac{1}{16} + \frac{1}{16} - \frac{1}{2048} - \frac{1}{1024} - \frac{1}{2048}$$

$$= \frac{8192 + 128 + 128 - 1 - 2 - 1}{2048}$$

$$= \frac{2111}{512}$$

Quadratic approximation: $\sqrt{17} \approx \frac{2111}{512} = 4.123046875$

Calculator: $\sqrt{17} = 4.1231056 \dots$

Error: $\left| \sqrt{17} - \frac{2111}{512} \right| < 0.00006$.

(*) **Looking ahead...**

(*) We won't be computing specific quadratic Taylor polynomials for approximation (or any other) purposes.

(*) We **will** be using the quadratic Taylor polynomial in two variables to understand the *second derivative test* in two variables. To this end, the key observation is the following.

If $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ (we say that (x_0, y_0) is a *critical point* in this case), then the quadratic Taylor polynomial for $f(x, y)$ centered at (x_0, y_0) is

$$T_2(x, y) = f(x_0, y_0) + \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0)$$
$$+ \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2$$

It follows in this case that if (x, y) is close to (x_0, y_0), then

$$f(x, y) - f(x_0, y_0) \approx T_2(x, y) - f(x_0, y_0)$$
$$= \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2$$

This observation is the foundation of the second derivative test in two variables.

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