(*) **Using the table of integration formulas**

The table in Appendix B (in the 13th edition of our textbook) contains 48 integration formulas. Most of these formulas are found using simple substitutions and so you could figure them out by yourself, but it is convenient and saves a lot of time to have them already done and listed. So we use the table.

On the other hand, several of the formulas are derived using other techniques, like *partial fraction decomposition* (described in Section 15.2) and *integration by parts* (described in Section 15.1).

Having access to a table of formulas doesn't mean that you can forget everything else — you will still frequently need to use algebra and substitution to simplify integrands before you can refer to the table.

Example 1. Formula #5 in the appendix is

$$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C.$$

In this (and all the other formulas in the appendix), u is the variable of integration and a and b are parameters (a fancy word for constants in this case). Furthermore, it assumed implicitly here that $a \neq 0$, otherwise the formula is wrong.[†]

Using this formula in the simplest case amounts to recognizing that it is the correct formula to use and plugging in the correct values for a and b. For example, to compute the following (definite) integral, we recognize that $u = x$, $a = 3$ and $b = 4$ and carefully apply formula #5 (and the FTC):

$$\begin{aligned}\int_1^5 \frac{2}{x(3+4x)}\,dx &= 2\int_1^5 \frac{dx}{x(3+4x)} \\ &= 2\left[\frac{1}{3}\ln\left|\frac{x}{3+4x}\right|\Bigg|_1^5\right] \\ &= \frac{2}{3}(\ln(5/23) - \ln(1/7)) = \frac{2}{3}\ln(35/23).\end{aligned}$$

(Using arithmetic properties of $\ln x$: $\ln(5/23) - \ln(1/7) = \ln\left(\frac{5/23}{1/7}\right) = \ln(35/23)$.)

Formula #5 is derived using the partial fraction decomposition

$$\frac{1}{u(a+bu)} = \frac{1/a}{u} - \frac{b/a}{a+bu} \implies \int \frac{du}{u(a+bu)} = \int \frac{1/a}{u}\,du - \int \frac{b/a}{a+bu}\,du$$

and the substitution $v = a + bu$, $du = \frac{1}{b}dv$ to compute

$$\int \frac{b/a}{a+bu}\,du = \frac{1}{a}\int \frac{dv}{v} = \frac{1}{a}\ln|v| + C = \frac{1}{a}\ln|a+bu| + C.$$

[†]If $b = 0$ here, the formula is correct. How?

(*) **Integration by parts**

Some of the formulas that appear in the table are found using a technique called *integration by parts*. This technique is based on the *product rule of differentiation*, which states that

$$\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)g(x) + f(x)g'(x),$$

and it follows that

$$\int f'(x)g(x) + f(x)g'(x)\,dx = f(x)g(x) + C.$$

Writing the integral on the left as a sum of two integrals and subtracting $\int f(x)g'(x)\,dx$ from bother sides leads to the formula for integration by parts:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

For this formula to be useful we have to be able to recognize that the integrand in a given integral is a product of the form $f'(x)g(x)$ *and* the integral on the right $\int f(x)g'(x)\,dx$ should be simpler (or at least as simple) as the integral we began with.

Example 2. Suppose that we want to compute $\int ue^{au}\,du$. The integrand is a product of two simple functions, i.e., neither factor is *composite* so it is unlikely that substitution will work here and the next thing to try is integration by parts.[‡]

If we set $g(u) = u$ and $f'(u) = e^{au}$, then $g'(u) = 1$ and $f(u) = \frac{1}{a}e^{au}$, so using integration by parts gives

$$\int ue^{au}\,du = \frac{1}{a}e^{au}\cdot u - \int \frac{1}{a}e^{au}\,du = \frac{ue^{au}}{a} - \frac{e^{au}}{a^2} + C = \frac{e^{au}}{a^2}\big(au-1\big) + C.$$

This is formula #38 in Appendix B. Formulas #38 through #43 are all found using integration by parts.

Example 3. Compute the definite integral

$$\int_0^{10} 3x^2e^{2x}\,dx.$$

First we apply formula #39,

$$\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du$$

with $n = 2$ and $a = 2$, and then we use formula #38 on the integral on the right:

$$\begin{aligned}\int_0^{10} 3x^2e^{2x}\,dx &= 3\int_0^{10} x^2e^{2x}\,dx = 3\left[\frac{x^2e^{2x}}{2}\bigg|_0^{10} - \frac{2}{2}\int_0^{10} xe^{2x}\,dx\right]\\ &= 3\left[\left(\frac{100e^{20}}{2} - 0\right) - \left(\frac{e^{2x}}{4}(2x-1)\bigg|_0^{10}\right)\right] = 3\left(50e^{20} - \frac{19}{4}e^{20} - \frac{1}{4}\right) = \frac{543e^{20}-3}{4}.\end{aligned}$$

[‡]Actually, in our course, the next thing to try is the table of integrals.

Example 4. Sometimes you have to make a substitution before to get an integral that you recognize in the table. Below, the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$ transforms the given integral to an integral to which formula #15 from the Appendix (with $a = 4$ and $b = 7$) may be applied.

$$\int \frac{3\ln x}{x\sqrt{4+7\ln x}}\,dx = 3\int \frac{u\,du}{\sqrt{4+7u}} = \frac{6(7u-8)\sqrt{4+7u}}{147} = \frac{6(7\ln x-8)\sqrt{4+7\ln x}}{147} + C.$$