(*) **Present value of a future payment**

If we invest $P_0$ dollars today in an account with a fixed annual interest rate $r$, compounded continuously, then the value of this investment at time $t_j$ years in the future is given by

$$P_j = P_0 e^{rt_j}.$$

Conversely, the *present value* (i.e., the value today) of a payment $P_j$ to be made at time $t_j$ in the future is equal to the amount $P_0$ that would need to be invested today in order to have $P_j$ at time $t_j$. In this scenario, we assume a constant interest rate $r$ compounded continuously, so the equation above tells us that

$$PV(P_j) = P_0 = P_j e^{-rt_j}.$$

(*) **Present value of an annuity**

An *annuity* is a sequence $A = \{P_1, P_2, \ldots, P_j, \ldots, P_n\}$ of payments to be made at times $t_1 < t_2 < \cdots < t_j < \cdots < t_n$ in the future.[†] The present value of such an annuity is simply the sum of the present values of the different payments, i.e.,

$$PV(A) = \sum_{j=1}^{n} PV(P_j) = \sum_{j=1}^{n} P_j e^{-rt_j}, \tag{1}$$

where in this formula we assume a constant interest rate $r$ compounded continuously.

The present value of the annuity is the amount $PV(A)$ that would need to be invested today in an account earning interest rate $r$, to have the funds to make the payments $P_1, \ldots, P_n$ at the appointed times $t_1, \ldots, t_n$. For example, when someone wins a large lottery prize and takes the lump sum payment instead of 26 annual payments, the amount they receive is essentially the present value of those annual payments.[‡]

The same calculation can be applied to any sequence of future amounts of money. For example, if somebody has a business and projects monthly incomes of $P_1, P_2, \ldots, P_n$ for $n$ months into the future, then the present value of this future income can be computed using formula (1).

(*) **Present value of a *continuous* annuity/income stream**

Many income streams are more-or-less continuous, in the sense that money comes in all the time, not once a week or once a day, but minute by minute (or even more frequently).[§] We can quantify such an income stream with two functions,

$$F(t) = \text{accumulated amount at time } t$$

[†]Traditionally, annuities paid once a year, hence the name.

[‡]Typically, the 26 annual payments are not all equal. The initial payments are lower and the later ones are larger, which has the effect of lowering the present value.

[§]Think website that gets paid a small amount every time someone clicks on one of the links that they display, or a company like Amazon that is making sales continuously.

and

$$f(t) = F'(t) = \text{the payment } \textbf{\textit{rate}} \text{ at time } t.$$

The quantity $F(t)$ is the total amount of money that will stream in between time $t_0 = 0$ (now) and time $t > 0$ (in the future), and $f(t)$ is the rate at which money is coming in per unit time.

The *nominal value* of such an income stream (the total amount of money that will accumulate between times $t = 0$ and $t = T$) is simply

$$F(T) = \int_0^T f(t)\, dt,$$

as follows from the fundamental theorem of calculus, because $F'(t) = f(t)$ and we may assume that $F(0) = 0$, so that

$$\int_0^T f(t)\, dt = \left. F(t) \right|_0^T = F(T) - F(0) = F(T).$$

On the other hand the *present value* should be less than this because future money has a lower value than present money. This leads to the question: *how can we find the present value of this income stream?*

We proceed (as usual) by finding a sequence of approximations to the present value, then we take the limit of these approximations (if it exists) to be the present value of the income stream.

First, we divide the interval $[0, T]$ into $n$ subintervals,

$$[t_0, t_1],\ [t_1, t_2],\ \ldots, [t_{j-1}, t_j], \ldots, [t_{n-1}, t_n],$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, and consider the amount of income that streams in during each of these time subintervals.

The amount of income that streams in during the first time interval is $F(t_1) = F(t_1) - F(t_0)$. Likewise, the amount that streams in during the second time interval is $F(t_2) - F(t_1)$ and in the third time interval the total is $F(t_3) - F(t_2)$. In general, the amount of income that accumulates in the interval $[t_{j-1}, t_j]$ is

$$P_j = F(t_j) - F(t_{j-1}),$$

which is the total amount through time $t_j$ minus what already accumulated up to time $t_{j-1}$. If the time intervals are short enough, then all of the money that accumulates in one of them has more or less the same present value, and we conclude that the present value of the continuous income stream $F(t)$ can be approximated by the present value of the $n$ payments $P_1, P_2, \ldots, P_n$ made at times $t_1, t_2, \ldots, t_n$, which we can compute using formula (1). I.e., we conclude that

$$PV(F) \approx \sum_{j=1} PV(P_j) = \sum_{j=1}^{n} P_j e^{-rt_j} = \sum_{j=1}^{n} \big(F(t_j) - F(t_{j-1})\big)\, e^{-rt_j}. \tag{2}$$

Given the context, we know that we will eventually use a definite integral to compute the present value, but the sum on the right does not look like a *Riemann sum*, mostly because there are no $\Delta t_j$ factors in the terms of the sum. To remedy this, we recall that if $\Delta t_j = t_j - t_{j-1}$ is small enough, then

$$F(t_j) - F(t_{j-1}) \approx F'(t_{j-1})\Delta t_j = f(t_{j-1})\Delta t_j.$$

Substituting $f(t_{j-1})\Delta t_j$ for $F(t_j) - F(t_{j-1})$ in the sum on the righthand side of (2) gives the approximation

$$PV(F) \approx \sum_{j=1}^{n} f(t_{j-1})\Delta t_j e^{-rt_j} \approx \sum_{j=1}^{n} f(t_{j-1}) e^{-rt_{j-1}} \Delta t_j.$$

The sum on the right above differs from the sum in the middle above, in that I replaced the factors $e^{-rt_j}$ by the factors $e^{-rt_{j-1}}$. This makes the sum on the right a left hand sum, and is justified by the argument that if $\Delta t_j$ is small enough, the difference between $e^{-rt_j}$ and $e^{-rt_{j-1}}$ is negligible.

Finally, as $n \to \infty$, the sums $\sum_{j=1}^{n} f(t_{j-1})e^{-rt_{j-1}}\Delta t_j$ approach a definite integral which we declare to be the present value we seek. I.e.,

$$PV(F) = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1})e^{-rt_{j-1}}\Delta t_j = \int_0^T f(t)e^{-rt}\, dt.$$

**Example.** Find the present value of a the income stream that pays at the rate $f(t) = 500t$ for $T = 20$ years, where the interest rate is $r = 3\%$.

Using the formula above (and formula # 38 in Appendix B, with $a = -0.03$), we have

$$PV = \int_0^{20} 500te^{-0.03t}\, dt = 500 \left[ \frac{e^{-0.03t}}{(-0.03)^2}(-0.03t - 1)\bigg|_0^{20} \right]$$

$$= \frac{500}{0.0009}\left(e^{-0.6}(-1.6) - e^0(-1)\right)$$

$$= \frac{500}{0.0009}\left(1 - 1.6e^{-0.6}\right) \approx 67722.99.$$