(*) **Separable differential equations**

In its most general form, a (first order, ordinary) differential equation is an equation of the form

$$\Phi(y, y', x) = 0,$$

whose solutions are functions $y = f(x)$. In general these equations can be very difficult to solve explicitly and frequently people use computers and numerical algorithms to generate approximate solutions.

On the other hand, certain differential equations can be relatively easy to solve, among these are ***separable*** differential equations. These are differential equations which can be put in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

This type of equation is called *separable* because multiplying the equation by $h(y) \cdot dx$ results in an equation where the variables x and y have been *separated*, an expression that equates two *differentials*,

$$h(y)\, dy = g(x)\, dx,$$

(hence the name *differential equation*). Integrating both sides of the separated equation leads to an *algebraic* equation,

$$\int h(y)\, dy = \int g(x)\, dx \qquad \Longrightarrow \qquad H(y) = G(x) + C,$$

which is called an *implicit solution* of the original differential equation (because it implies a relation between y and x). Solving the algebraic equation for y results in an explicit solution,

$$y = H^{-1}(G(x) + C),$$

or more accurately, a family of solutions, one for each possible value of C. Finally, given *data* in the form $y(x_0) = y_0$, we can solve for C and find the unique solution $y = f(x)$ of the *initial value problem*

$$y' = \frac{g(x)}{h(y)} \quad \text{and} \quad y(x_0) = y_0.$$

(It can be shown that if $g(x)$ is continuous in an interval around x_0 and $h(y)$ is continuous and $h(y) \neq 0$ in an interval around y_0, then a unique solution does exist. Finding it easily is another question.)

Example 1. Find the function $y = f(x)$ satisfying

$$y' = \frac{2x+1}{y+2} \quad \text{and} \quad y(1) = 1.$$

Step 1. Separate:

$$y' = \frac{2x+1}{y+2} \Longrightarrow \frac{dy}{dx} = \frac{2x+1}{y+2} \Longrightarrow y + 2\, dy = 2x + 1\, dx.$$

Step 2. Integrate:

$$\int y + 2\,dy = \int 2x + 1\,dx \implies \frac{y^2}{2} + 2y = x^2 + x + C.$$

Step 3. Solve for y:

$$\frac{y^2}{2} + 2y = x^2 + x + C \implies y^2 + 4y = 2x^2 + 2x + C \implies y^2 + 4y - \left(2x^2 + 2x + C\right) = 0$$

The last equation on the right is a quadratic equation in y of the from $ay^2 + by + c = 0$, where $a = 1$, $b = 4$ and $c = -\left(2x^2 + 2x + C\right)$, and we can solve it using the quadratic formula:

$$\begin{aligned}
y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-4 \pm \sqrt{16 + 4\left(2x^2 + 2x + C\right)}}{2} \\
&= \frac{-\cancel{4}^{\,2} \pm \cancel{2}\sqrt{4 + 2x^2 + 2x + C}}{\cancel{2}} \quad \text{(factoring 4 out of the } \sqrt{\ }\text{)} \\
&= -2 \pm \sqrt{2x^2 + 2x + C} \quad \text{(`absorbing' the term 4 into the constant } C\text{)}
\end{aligned}$$

So $y = -2 \pm \sqrt{2x^2 + 2x + C}$ and to solve for C and determine whether the '$\pm$' is $+$ or $-$, we use the data, $y(1) = 1$.

First, since $y(1) = 1 > 0$, we must choose the $+$ sign, otherwise $y < 0$ when $x = 1$. This means that

$$1 = y(1) = -2 + \sqrt{2 \cdot 1^2 + 2 \cdot 1 + C} \implies \sqrt{4 + C} = 3 \implies C = 5,$$

so the (unique) solution to this initial value problem is

$$y = \sqrt{2x^2 + 2x + 5} - 2.$$

Exercise: Compute y' and verify that it satisfies the differential equation.

(*) **Elasticity**

Given a functional relation $y = f(x)$, the x-elasticity of y is defined as

$$\eta_{y/x} = \lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x} = \frac{dy}{dx} \cdot \frac{x}{y},$$

where $\%\Delta y$ and $\%\Delta x$ are the percentage-changes in y and x, respectively.

Economic theory will sometimes dictate that the x-elasticity of y has certain characteristics, and these characteristics can lead to a differential equation for the (unknown) function $y = f(x)$.

Example 2. A short term production function

$$Q = P(L)$$

is one where the capital input (K) is considered to be fixed while the labor input (L) is variable. The variable Q is output.

In some contexts, economists will specify that the *labor-elasticity of output* is constant. The condition $\eta_{Q/L} = \beta$, where β is the (typically unknown) constant value of of the elasticity, leads to the *separable* differential equation

$$\frac{dQ}{dL} \cdot \frac{L}{Q} = \beta \qquad \Longrightarrow \qquad \frac{dQ}{Q} = \beta \cdot \frac{dL}{L}.$$

Integrating both sides

$$\int \frac{dQ}{Q} = \beta \int \frac{dL}{L},$$

leads to the implicit relation

$$\ln Q = \beta \ln L + C.$$

To solve for Q, we exponentiate both sides,

$$e^{\ln Q} = e^{\beta \ln L + C} = e^C \cdot e^{\ln(L^\beta)} \qquad \Longrightarrow \qquad Q = AL^\beta,$$

where $A = e^C$. In other words, if the labor elasticity of output is constant then the short term production function is a *power function*, where the power is equal to the constant elasticity.†

This example illustrates another common feature of differential equations, namely that they might include unspecified parameters. The specific solution of such an equation might require more than one data point.

Example 2. (continued) Suppose that a short term production function $Q = P(L)$ has constant labor-elasticity of output, then we know that

$$Q = AL^\beta$$

but we can't know the values of A or β without more information. Since there are two unknown parameters, it stands to reason that we will need two data points, so suppose that when labor input is $L_0 = 100$, the output is $Q_0 = 1000$ and when labor input is $L_1 = 200$, the output is $Q_1 = 1500$. This data, together with the relation $Q = AL^\beta$ leads to the pair of equations below for A and β:

$$1000 = A(100)^\beta \tag{1}$$

$$1500 = A(200)^\beta \tag{2}$$

To solve this pair of equations, we can (i) take logarithms of both sides, which converts them into the following pair of linear equations (for β and $\ln A$)

$$\begin{aligned} \ln 1000 &= \ln A + \beta \ln 100 \\ \ln 1500 &= \ln A + \beta \ln 200, \end{aligned}$$

† The conclusion is of course general — if $\eta_{y/x} = \beta$, then $y = Ax^\beta$, where A is another constant — it doesn't matter what the variables x and y represent.

which we know how to solve,[‡] or we can (ii) proceed as follows.

The quotient of the lefthand side of (2) by the lefthand side of (2) is equal to the quotient of the righthand side of (2) by the righthand side of (2):

$$\frac{1500}{1000} = \frac{\cancel{A}(200)^{\beta}}{\cancel{A}(100)^{\beta}} \implies 1.5 = 2^{\beta},$$

This has the effect of eliminating the parameter A. Now, taking logarithms of both sides of the equation on the right and moving things around, gives the value of β:

$$\ln 1.5 = \beta \ln 2 \implies \beta = \frac{\ln 1.5}{\ln 2} \quad (\approx 0.585).$$

Finally, returning to (1) again, we can solve for A:

$$1000 = A(100)^{\beta} \implies A = 1000 \cdot (100)^{-\beta} = 1000 \cdot (100)^{-\ln 1.5/\ln 2} \approx 67.62.$$

So the production function is

$$Q \approx 67.22 L^{0.585}.$$

Exercise. Solve the pair of linear equations at the bottom of the last page and check that you obtain the same values for A and β as above.

In many cases, elasticity is not constant. This is certainly common for price-elasticity of demand, where the elasticity changes as the price changes, i.e., where the price-elasticity of demand is a function of the price. Happily,[§] these assumptions also lead to separable differential equations

Example 3. The price-elasticity of demand q for a certain good is assumed to be *proportional to* the square root of the price p of that good. When the price is $p_0 = 9$, the demand is $q_0 = 500$ and when the price is $p_1 = 25$, the demand is $q_1 = 300$.

What will demand be when the price is $p_2 = 36$?

The phrase *'proportional to'* means *'multiple of'*, so that the assumption above about the price-elasticity of demand leads to the separable differential equation

$$\eta_{q/p} = k\sqrt{p} \implies \frac{dq}{dp} \cdot \frac{p}{q} = k\sqrt{p} \implies \frac{dq}{q} = k\frac{\sqrt{p}}{p}\, dp = kp^{-1/2}\, dp,$$

where k is the (unknown) constant of proportionality. Integrating both sides of the equation on the right yields an implicit relation between q and p,

$$\int \frac{dq}{q} = k \in p^{-1/2}\, dp \implies \ln q = k \cdot \frac{p^{1/2}}{1/2} + C = k_1 p^{1/2} + C,$$

where $k_1 = 2k$ is still an unknown constant. As in the previous example, we solve for q by exponentiating both sides of the last equation,

$$e^{\ln q} = e^{k_1 p^{1/2} + C} = e^{C} \cdot e^{k_1 p^{1/2}} \implies q = Ae^{k_1 p^{1/2}},$$

[‡]Right?

[§]Depending on your perspective.

where (also as before) $A = e^C$.

Once again, we now use the data to solve for A and k_1, using the two equations $q_0 = Ae^{k_1\sqrt{p_0}}$ and $q_1 = Ae^{k_1\sqrt{p_1}}$:

$$500 = Ae^{k_1\sqrt{9}} = Ae^{3k_1}$$
$$300 = Ae^{k_1\sqrt{25}} = Ae^{5k_1}$$

Dividing left side by left side and right side by right side again eliminates the A (again),

$$\frac{300}{500} = \frac{\cancel{A}e^{5k_1}}{\cancel{A}e^{3k_1}} \implies 0.6 = e^{5k_1-3k_1} = e^{2k_1},$$

and taking logarithms again gives

$$\ln 0.6 = 2k_1 \implies k_1 = \frac{\ln 0.6}{2} \quad (\approx -0.2554).$$

Next, using the first data point (or the second) we find that

$$500 = Ae^{3k_1} \implies A = 500e^{-3k_1} \quad (\approx 1075.83),$$

and finally, when the price is $p_2 = 36$, the demand will be

$$q_2 = Ae^{k_1\sqrt{36}} = 500e^{-3k_1}e^{6k_1} = 500e^{3k_1} = 500e^{\frac{3}{2}\ln 0.6} \approx 232.38.$$