(*) **Examples: Profit Maximization**

Joint weekly demand functions for a firm's competing products:

$$Q_A = 100 - 3P_A + 2P_B$$
$$Q_B = 60 + 2P_A - 2P_B$$

Weekly cost of producing Q_A units of product A and Q_B units of product B:

$$C = 20Q_A + 30Q_B + 1200$$

Firm's weekly profit function

$$\begin{aligned}\Pi &= P_AQ_A + P_BQ_B - C\\ &= P_A(100 - 3P_A + 2P_B) + P_B(60 + 2P_A - 2P_B)\\ &\quad - (20(100 - 3P_A + 2P_B) + 30(60 + 2P_A - 2P_B) + 1200)\\ &= -3P_A^2 + 4P_AP_B - 2P_B^2 + 100P_A + 80P_B - 5000\end{aligned}$$

Weekly profit function

$$\Pi = -3P_A^2 + 4P_AP_B - 2P_B^2 + 100P_A + 80P_B - 5000$$

First order conditions for max:

$$\begin{aligned}\Pi_{P_A} = 0 &\quad\Longrightarrow\quad & -6P_A + 4P_B + 100 = 0\\ \Pi_{P_B} = 0 &\quad\Longrightarrow\quad & 4P_A - 4P_B + 80 = 0\end{aligned}$$

Adding the two equations together gives an equation for the critical P_A value:

$$-2P_A + 180 = 0 \implies P_A^* = 90.$$

Substituting this in the second equation ($\Pi_{P_B} = 0$) yields the critical P_B value:

$$4 \cdot 90 - 4P_B + 80 = 0 \implies -4p_B + 440 = 0 \implies P_B^* = 110.$$

The corresponding critical weekly outputs are

$$Q_A^* = 100 - 3P_A^* + 2P_B^* = 50 \quad \text{and} \quad Q_B^* = 60 + 2P_A^* - 2P_B^* = 20.$$

The critical weekly revenue is $R^* = P_A^*Q_A^* + P_B^*Q_B^* = 6700$,

the critical weekly cost is $C^* = 20Q_A^* + 30Q_B^* + 1200 = 2800$,

and the critical weekly profit is $\Pi^* = R^* - C^* = 3900$.

The critical question: *Is Π^* the **maximum** weekly profit?*

To answer this, we need...

(*) **The second derivative test**

The second derivative test for a function of two variables states the following:

Suppose that $f_x(a,b) = 0 = f_y(a,b)$ (so (a,b) is a critical point) and let

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

*be the **discriminant**. Then there are three possibilities for $D(a,b)$.*

D1. *If $D(a,b) > 0$, then $f(a,b)$ is either a relative minimum value or a relative maximum value, depending on $f_{xx}(a,b)$ (or equivalently, $f_{yy}(a,b)$):*

- *If $f_{xx}(a,b) > 0$, then $f(a,b)$ is a relative minimum value.*
- *If $f_{xx}(a,b) < 0$, then $f(a,b)$ is a relative maximum value.*

D2. *If $D(a,b) < 0$, then $f(a,b)$ is neither a relative minimum value nor a relative maximum value.*

D3. *If $D(a,b) = 0$, then we cannot draw any conclusions about $f(a,b)$.*

Profit maximization example (continued):

We found the critical prices to be $P_A^* = 90$ and $P_B^* = 110$ and the critical profit to be $\Pi^* = 3900$. The second derivatives in this case are

$$\Pi_{P_AP_A} = -6, \quad \Pi_{P_AP_B} = 4 \text{ and } \Pi_{P_BP_B} = -4,$$

so the discriminant is

$$D(P_A^*, P_B^*) = \Pi_{P_AP_A} \cdot \Pi_{P_BP_B} - \Pi_{P_AP_B}^2 = 24 - 16 = 8 > 0.$$

Since the discriminant is positive and $\Pi_{P_AP_A} = -6 < 0$, the critical profit $\Pi^* = 3900$ is a maximum value.

(*) **Justification of the second derivative test.**†

The second derivative test is based on the following observation:

If (a,b) is a critical point of the function $f(x,y)$, then

$$f(x,y) \approx f(a,b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

for all points (x,y) that are close to (a,b).‡

Starting from this observation, we first subtract $f(a,b)$ from both sides of the approximate identity above to see that if (x,y) is close to (a,b), then

$$f(x,y) - f(a,b) \approx Au^2 + Buv + Cv^2, \tag{1}$$

where (to make writing things easier)

†Algebra Will Robinson! Algebra Will Robinson!!

‡This follows from quadratic approximation, as you can read about in SN 2 on the Supplements page of the course website.

- $A = \dfrac{f_{xx}(a,b)}{2}$, $B = f_{xy}(a,b)$ and $C = \dfrac{f_{yy}(a,b)}{2}$;
- $u = (x-a)$ and $v = (y-b)$.

Next, we pull out a factor of A from the right hand side of (1) and then *complete the square* in the bracketed expression:

$$\begin{aligned}
Au^2 + Buv + Cv^2 = A\left[u^2 + \frac{B}{A}uv + \frac{C}{A}v^2\right] &= A\left[\underbrace{u^2 + \frac{B}{A}uv + \frac{B^2}{4A^2}v^2} - \frac{B^2}{4A^2}v^2 + \frac{C}{A}v^2\right] \\
&= A\left[\left(u + \frac{B}{2A}v\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)v^2\right] \\
&= A\left[\left(u + \frac{Bv}{2A}\right)^2 + \left(\frac{4AC - B^2}{4A^2}\right)v^2\right] \\
&= A\left[\left(u + \frac{Bv}{2A}\right)^2 + D(a,b)\left(\frac{v}{2A}\right)^2\right],
\end{aligned}$$

where

$$D(a,b) = 4AC - B^2 = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2.$$

Finally, ignoring the algebraic wizardry in the last paragraph, we conclude that if (x,y) is close to the critical point (a,b), then

$$f(x,y) - f(a,b) \approx A\overbrace{\left[\left(u + \frac{Bv}{2A}\right)^2 + D\cdot\left(\frac{v}{2A}\right)^2\right]}^{Q(x,y)}.$$

The squared expressions in $Q(x,y)$ are always positive, so if $D(a,b)$ is also positive, then $Q(x,y) > 0$ for all (x,y). In this case, there are two possibilities:

- If $A = f_{xx}(a,b) > 0$, then $f(x,y) - f(a,b) > 0$ for all (x,y) close to (a,b), which means that $f(a,b)$ is a relative minimum value.
- If $A = f_{xx}(a,b) < 0$, then $f(x,y) - f(a,b) < 0$ for all (x,y) close to (a,b), which means that $f(a,b)$ is a relative maximum value.

This case with its two subcases corresponds to case **D1** of the second derivative test.

On the other hand, if $D(a,b) < 0$ then $Q(x,y)$ has the form $S^2 - |D|T^2$, which will assume positive values at some points (x,y) close to (a,b) and negative values at other points (x,y) close to (a,b). This means that $AQ(x,y)$ will also assume both positive and negative values, and therefore the difference $f(x,y) - f(a,b)$ will be positive for some points close to (a,b) and negative at others. Therefore, $f(a,b)$ is neither a minimum nor a maximum value of the function, since there are nearby values which are greater and other nearby values which are smaller. This corresponds to case **D2** of the second derivative test.

The explanation for case **D3**, is a little more involved, so we will leave it alone.

(*) **Pretty pictures**

In the vicinity of a relative maximum or minimum value (i.e., case **D1** of the secoond derivative test) the graph of a function $z = f(x, y)$ will have one of the shapes below.

Figure 1: Case **D1** of the second derivative test, both possibilities.

In the vicinity of a critical point where the discriminant is negative (i.e., case **D2** of the secoond derivative test) the graph of a function $z = f(x, y)$ will often (but not always) have a shape similar to the one below.

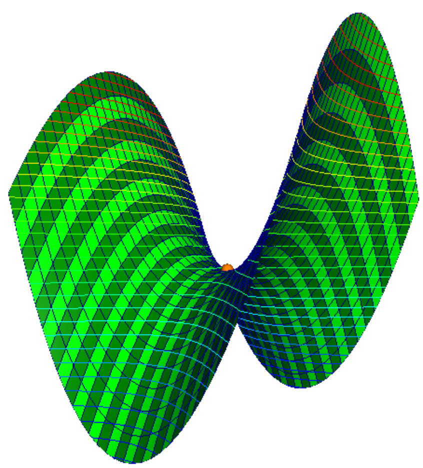

Figure 2: Case **D2** of the second derivative test.

This shape is reminiscent of a western saddle, and a point like this called a saddle point for this reason.

(*) **Example.**

Find the critical points and classify the critical values of

$$f(x, y) = x^2 + y^2 - xy + x^3.$$

The partial derivatives are

$$f_x = 2x - y + 3x^2 \quad \text{and} \quad f_y = 2y - x.$$

and solving the pair of equations

$$\begin{aligned} 2x - y + 3x^2 &= 0 \\ 2y - x &= 0 \end{aligned}$$

we find that the critical points are $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (-1/2, -1/4)$, with critical values $f(0, 0) = 0$ and $f(-1/2, -1/4) = 1/16$.

On to the second derivative test:

Discriminant: $f_{xx} = 2 + 6x$, $f_{xy} = -1$ and $f_{yy} = 2$, so

$$D(x, y) = \overbrace{2(2 + 6x)}^{f_{xx} f_{yy}} - \overbrace{(-1)^2}^{f_{xy}^2} = 12x + 3.$$

Analysis:

(*) $D(0, 0) = 3 > 0$ and $f_{xx}(0, 0) = 2 > 0$, so $f(0, 0) = 0$ is a *relative minimum value.*

(*) $D(-1/2, -1/4) = -3 < 0$, so $f(-1/2, -1/4) = 5/16$ is *neither a minimum nor a maximum value.*

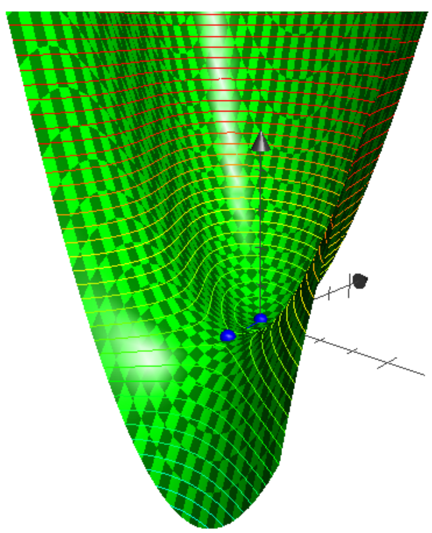

Figure 3: Graph of $z = x^2 + y^2 - xy + x^3$, with critical points (blue dots)