(*) **The Envelope Theorem and the multiplier in constrained optimization**

As we saw in class on Wednesday (and in Supplemental Notes #5), one of the important *applications* of the envelope theorem is to constrained optimization.

Suppose that we solve the problem of finding the maximum/minimum value of the (objective) function $f(x, y, z)$ subject to a constraint of the form $g(x, y, z) = c$ using the method of Lagrange multipliers. Briefly, we form the Lagrangian

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda(g(x, y, z) - c)$$

and find the critical point $(x^*, y*, z^*, \lambda^*)$ by solving the system of equations

$$F_x = 0, \quad F_y = 0, \quad F_z = 0 \quad \text{and} \quad F_\lambda = 0.$$

Since $F_\lambda = -(g(x, y, z) - c)$, it follows that $g(x^*, y^*, z^*) - c = 0$, and therefore

$$F^* = F(x^*, y^*, z^*, \Lambda^*) = f(x^*, y^*, z^*) - \lambda^* \overbrace{(g(x^*, y^*, z^*) - c)}^{=0} = f(x^*, y^*, z^*) = f^*.$$

That is to say the constrained optimum f^* is always the same as the critical value of the Lagrangian F^*.

This is useful, because it allows us to easily find the rate of change of f^* with respect to the constraint c. Specifically, we can use the envelope theorem as follows.

$$\overbrace{\frac{df^*}{dc} = \frac{dF^*}{dc}}^{f^*=F^*} = \overbrace{\frac{\partial F}{\partial c}\bigg|_{\substack{x=x^*\\ y=y^*\\ z=z^*\\ \lambda=\lambda^*}}}^{\text{Envelope Theorem}} = \lambda\bigg|_{\substack{x=x^*\\ y=y^*\\ z=z^*\\ \lambda=\lambda^*}} = \lambda^*$$

because

$$\frac{\partial F}{\partial c} = \frac{\partial}{\partial c}\big(f(x, y, z) - \lambda(g(x, y, z) - c)\big) = \frac{\partial}{\partial c}\big(f(x, y, z) - \lambda g(x, y, z) + \lambda c\big) = \lambda.$$

In words, *the critical value of the multiplier λ^* gives the rate of change of the constrained optimum f^* with respect to the constraining parameter c.* In practical terms, linear approximation tells us that if the constraining parameter changes by Δc, then the constrained optimum changes by $\Delta f^* \approx (df^*/dc) \cdot \Delta c = \lambda^* \cdot \Delta c$. In particular, if $\Delta c = 1$, then $\Delta f^* \approx \lambda^*$.

(*) **But...**

The interpretation of λ^* given above is just one (important) *application* of the envelope theorem. ***The envelope theorem is a general tool for finding the rate of change of a critical value with respect to a parameter***, and shouldn't be confused with one of its applications.