Constrained optimization.

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1.   Constraints

In many of the optimization problems that arise in economics, there are restrictions on the values that the independent variables may take.

Example 1.   A firm's cost function is given by

$$C = 0.05Q_A^2 + 0.01Q_A Q_B + 0.03Q_B^2 + 10Q_A + 15Q_B + 12000, \tag{1}$$

where Q_A and Q_B are the quantities of the firm's product that are produced in the firm's two facilities A and B, respectively. If the firm has a contract to produce 2000 units of output, then how many units should they produce in each facility to *minimize* their cost?

The condition that total output should equal 2000, imposes the following restriction on the variables Q_A and Q_B

$$Q_A + Q_B = 2000, \tag{2}$$

together with the conditions $Q_A, Q_B \geq 0$, since neither output can be negative.   ■

The condition in equation (2) is called a ***constraint*** because it *constrains* (restricts) the possible values of the free variables Q_A and Q_B. The function to be optimized (the cost function, in the example above) is called the ***objective function*** in these problems.

Example 2.   A firm's output is given by the Cobb-Douglas model

$$Q = AK^\alpha L^\beta, \tag{3}$$

where Q is the firm's output, K is quantity of the firm's capital input and L is the quantity of the firm's labor input. The constants, α, β and A are all positive and we also assume that $\alpha + \beta = 1$. If the prices per unit of capital and labor are p_K and p_L, respectively, and the firm's production budget is B, then *how should the firm allocate its budget to maximize its output?*

The constraint in this case is given by the equation

$$p_K \cdot K + p_L \cdot L = B, \tag{4}$$

reflecting the facts that (a) it costs $p_K \cdot K + p_L \cdot L$ to use K units of capital and L units of labor, and (b) the total cost must equal B. The objective function in this example is the output, Q.   ■

In what follows, we'll see two approaches to solving this type of optimization problem. One approach, substitution, is more elementary, and uses the constraint to reduce the number of variables. The second approach, *the method of Lagrange multipliers*, is more sophisticated and actually introduces a new variable to the problem.[†] On the other hand,

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[†]Or more than one new variable, if there is more than one constraint.

this additional variable yields important information about the optimization problem that is more difficult to derive using the substitution method. Also, it is often the case, that the algebra involved in finding the relevant critical points is actually easier in the Lagrange multiplier approach.

2.   Substitution

One approach to solving a constrained optimization problem is to use the constraint (or constraints, if there is more than one) to reduce the number of variables, and transform the problem to an **unconstrained optimization problem in fewer variables**.

Example 3.   To illustrate this, let's solve the cost minimization problem in Example 1.

First, we use the constraint in Equation (2) to express the variable Q_B in terms of the variable Q_A:

$$Q_B = 2000 - Q_A.$$

Next, substitute this expression into the objective function in (1) for Q_B, to express the cost function as a function of Q_A alone:

$$C(Q_A) = 0.05Q_A^2 + 0.01Q_A(2000 - Q_A) + 0.03(2000 - Q_A)^2 \tag{5}$$
$$+ 10Q_A + 15(2000 - Q_A) + 12000.$$

Since both Q_A and Q_B must be nonnegative and since their sum is 2000, it follows that $0 \le Q_A \le 2000$, and so we have a simple *find-the-minimum-on-a-closed-interval* problem. Simplifying the cost function in (5), we find that we need to minimize the function

$$C(Q_A) = 0.07Q_A^2 - 105Q_A + 162000$$

on the interval $[0, 2000]$. Differentiating and setting equal to 0 gives

$$C'(Q_A) = 0.14Q_A - 105 = 0,$$

yielding one critical point, $Q_A^* = 750$. The critical value $C^* = C(750) = 122625$ is the minimum value on $[0, 2000]$, as you can verify by comparing to the values at the endpoints.[‡] We conclude that the firm should produce 750 units in facility A and 1250 units in facility B to minimize their cost of production. ∎

This approach can be relatively simple, as in the previous example, and it can be a bit more complicated. As an exercise, you should try the substitution method to solve the constrained problem in Example 9. We can also use this approach in the case that the objective function has three or more variables.

Example 4.   A household's *utility* function is given by

$$U(x, y, z) = 5 \ln x + 8 \ln y + 12 \ln z, \tag{6}$$

where x, y and z are the monthly quantities of goods of types 1, 2 and 3, respectively, that the family consumes in a month. The average price per unit of type 1 goods is $p_1 = \$10.00$; the average price per unit of type 2 goods is $P_2 = \$15.00$; the average price per unit of type 3 goods is $P_3 = \$30.00$; and the household's monthly disposable income (which it spends entirely on these goods) is $B = \$3000.00$.
How should the household allocate its budget to maximize its monthly utility?

[‡]In fact, this is the minimum value on $(-\infty, \infty)$. Why?

From the prices and income, we obtain the *income constraint*

$$10x + 15y + 30z = 3000, \tag{7}$$

since the amount on the left-hand side of the equation is the cost of consuming x, y and z units of goods of types 1, 2 and 3, and this must be equal to the disposable income, on the right.

Solving (7) for the variable x, we find that $x = 300 - 1.5y - 3z$, and substituting this expression in the utility function gives a function of the two variables y and z, (which I'll denote with a lower case u, to indicate that the constraint has been used to substitute for the variable x),

$$u(y, z) = 5\ln(300 - 1.5y - 3z) + 8\ln y + 12\ln z.$$

We now want to find the maximum value of this function without any constraints, except that the conditions $x = 300 - 1.5y - 3z > 0$, $y > 0$ and $z > 0$ must be satisfied (why?), which means that the points (y, z) we consider must lie in the *triangle* in the (y, z)-plane bounded by the lines $y = 0$, $z = 0$ and $y = 200 - 2z$.

The first order conditions are

$$u_y = -\frac{7.5}{300 - 1.5y - 3z} + \frac{8}{y} = 0$$

$$u_z = -\frac{15}{300 - 1.5y - 3z} + \frac{12}{z} = 0.$$

Putting the expressions in the two equations on common denominators gives the equations

$$-\frac{7.5y}{y(300 - 1.5y - 3z)} + \frac{8(300 - 1.5y - 3z)}{y(300 - 1.5y - 3z)} = \frac{2400 - 19.5y - 24z}{y(300 - 1.5y - 3z)} = 0$$

$$-\frac{15z}{z(300 - 1.5y - 3z)} + \frac{12(300 - 1.5y - 3z)}{z(300 - 1.5y - 3z)} = \frac{3600 - 18y - 51z}{z(300 - 1.5y - 3z)} = 0.$$

Finally, since the numerator of a simple ratio must be 0 for the ratio to equal 0, this last pair of equations reduces to a simple pair of linear equations

$$19.5y + 24z = 2400$$
$$18y + 51z = 3600.$$

Solving the first equation for z gives $z = 100 - \frac{19.5}{24}y$, and substituting this into the second equation and solving for y gives $y_0 = 64$. Plugging this back into the expression for z gives $z_0 = 100 - \frac{19.5}{24}y_0 = 48$. Thus we found one critical point, $(y_0, z_0) = (64, 48)$.

To verify that the critical point we found yields the *maximum* value that we seek, we need to check the *second-order* conditions. I.e., we need to check that $D = u_{yy}u_{zz} - u_{yz}^2 > 0$ and $u_{yy} < 0$ at the critical point $(y_0, z_0) = (64, 48)$.

We have

$$u_{yy} = -\left(\frac{11.25}{(300 - 1.5y - 3z)^2} + \frac{8}{y^2}\right),$$

$$u_{zz} = -\left(\frac{45}{(300 - 1.5y - 3z)^2} + \frac{12}{z^2}\right) \text{ and}$$

$$u_{yz} = -\left(\frac{22.5}{(300 - 1.5y - 3z)^2}\right),$$

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and evaluating at the critical point gives

$$D(64, 48) = 0.000050863 > 0 \quad \text{and} \quad u_{yy}(64, 48) = -0.005078125 < 0.$$

Thus $u(64, 48) \approx 100.197$ is indeed the maximum value we seek.

Returning to the original 3-variable problem, we find that $x_0 = 300 - 1.5y_0 - 3z_0 = 60$, and conclude that, subject to their income constraint, the household's utility is maximized when they consume 60 units of type 1 goods, 64 units of type 2 goods and 48 units of type 3 goods. ∎

3. Lagrange multipliers

The second approach for solving constrained optimization problems is based on considerations from *vector calculus*, and is named for the mathematician *Joseph Louis Lagrange*. This approach is more sophisticated, and as a consequence yields more information.

Suppose that we want to find the optimal value of the objective function $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$. The method of Lagrange multipliers introduces a new variable, traditionally denoted by the Greek letter λ ('lambda'), and the new function

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda \cdot \big(g(x, y, z) - c \big), \tag{8}$$

called the '*Lagrangian*'. The new variable λ is called the *multiplier*.[§] The principle behind the method is the following.

> *Under suitable conditions, the optimal value(s) of the objective function, $f(x, y, z)$, subject to the constraint, $g(x, y, z) = c$, is (are) obtained at the (x, y, z)-components of stationary point(s) $(x^*, y^*, z^*, \lambda^*)$ of the lagrangian $F(x, y, z, \lambda)$.*

The stationary points of the Lagrangian function are found by setting all of its first order partial derivatives equal to 0 (as usual) and solving the resulting system of equations,

$$\left. \begin{aligned} F_x(x, y, z, \lambda) &= f_x(x, y, z) - \lambda \cdot g_x(x, y, z) = 0, \\ F_y(x, y, z, \lambda) &= f_y(x, y, z) - \lambda \cdot g_y(x, y, z) = 0, \\ F_z(x, y, z, \lambda) &= f_z(x, y, z) - \lambda \cdot g_z(x, y, z) = 0, \\ &\text{and} \\ F_\lambda(x, y, z, \lambda) &= -(g(x, y, z) - c) = 0. \end{aligned} \right\} \tag{9}$$

Note that the fourth equation ($F_\lambda = 0$) in the system of equations above reproduces the original constraint because

$$-(g(x, y, z) - c) = 0 \quad \Longleftrightarrow \quad g(x, y, z) - c = 0 \quad \Longleftrightarrow \quad g(x, y, z) = c.$$

Moreover, this implies that if $(x^*, y^*, z^*, \lambda^*)$ is a critical point, then

$$F(x^*, y^*, z^*, \lambda^*) = f(x^*, y^*, z^*)$$

because $g(x^*, y^*, z^*) - c = 0$. In other words, the critical *values* of $F(x, y, z, \lambda)$ are equal to the corresponding values of $f(x, y, z)$ at the (x, y, z)-coordinates of the critical points.

[§]The plural, 'multipliers', is used in the name of the method because in principle, there may be more than one constraint. If there are k constraints, $g_1 = c_1, g_2 = c_2, \ldots, g_k = c_k$, then k multipliers are introduced, and the Lagrangian has the form $F = f - \lambda_1(g_1 - c_1) - \lambda_2(g_2 - c_2) - \cdots - \lambda_k(g_k - c_k)$.

The first three equations in the system (9) can often be rewritten in a form that reveals structural aspects of the model being studied,[¶] and for this reason I refer to them as the *structural equations* of the optimization problem. Another way of thinking of these equations is as *equilibrium equations* for the problem being studied.

Example 5. First, let's apply the method to a problem that we have already solved, e.g., the utility maximization problem in Example 4.

The objective function is the utility $U(x, y, z) = 5 \ln x + 8 \ln y + 12 \ln z$ and the (income) constraint is given by $10x + 15y + 30z = 3000$. Putting these together as in (8), we find that the Lagrangian function for this problem is

$$F(x, y, z, \lambda) = 5 \ln x + 8 \ln y + 12 \ln z - \lambda(10x + 15y + 30z - 3000).$$

The structural equations in this case are

$$\frac{5}{x} - 10\lambda = 0, \quad \frac{8}{y} - 15\lambda = 0, \quad \text{and} \quad \frac{12}{z} - 30\lambda = 0,$$

and solving each of these equations for λ produces the following 'triple' equation:

$$\lambda = \frac{1}{2x} = \frac{8}{15y} = \frac{2}{5z}. \tag{10}$$

Comparing the x-term to the y-term gives $15y = 16x$ which implies $y = \frac{16}{15}x$. Likewise, comparing the x-term to the z-term gives $5z = 4x$ which implies $z = \frac{4}{5}x$. Finally, inserting these expressions for y and z into the constraint gives

$$10x + 15\left(\frac{16}{15}x\right) + 30\left(\frac{4}{5}x\right) = 50x = 3000,$$

yielding one critical point: $(x^*, y^*, z^*, \lambda^*) = (60, 64, 48, 1/120)$.

The first thing to note is that $(x^*, y^*, z^*) = (60, 64, 48)$ is the same critical point that we found using the substitution method, and which we know produces the maximum that we seek (based on the analysis in Example 4). Second, the algebra was easier using the method of Lagrange multipliers than using the method of substitution.

Finally, we have also produced a new piece of information, namely the critical value of the multiplier, $\lambda^* = 1/120$. As we will see, the critical value of the multiplier has an important mathematical interpretation, which leads to important economic interpretations. We'll get to that when we study the *envelope theorem*. ∎

Example 6. Studying the utility maximization problem more generally (and more theoretically) leads to a better understanding of the multiplier. We can also draw economic conclusions, and interpret the structural equations as *equilibrium* conditions for utility maximization.

Let's denote the utility function by $U(q_1, q_2, q_3)$, where the variables q_1, q_2 and q_3 are the quantities of goods of types 1, 2 and 3 that are consumed per unit time, respectively. As before, each type of good has an average price, which I'll denote here by p_1, p_2 and p_3 respectively, all measured in dollars, for concreteness' sake. This leads to the income constraint

$$p_1 \cdot q_1 + p_2 \cdot q_2 + p_3 \cdot q_3 = Y_d, \tag{11}$$

where Y_d is the consumer's disposable income.

[¶]I.e., they describe some aspect of the (economic) theory behind the optimization problem.

It follows that the Lagrangian for the utility maximization problem is

$$F(q_1, q_2, q_3, \lambda) = U(q_1, q_2, q_3) - \lambda(p_1 q_1 + p_2 q_2 + p_3 q_3 - Y_d),$$

and the structural equations may be written as

$$\begin{aligned}
U_{q_1}(q_1, q_2, q_3) &= \lambda p_1 \\
U_{q_2}(q_1, q_2, q_3) &= \lambda p_2 \\
U_{q_3}(q_1, q_2, q_3) &= \lambda p_3.
\end{aligned}$$

Solving each of these equations for λ leads to the following triple equation

$$\lambda = \frac{U_{q_1}(q_1, q_2, q_3)}{p_1} = \frac{U_{q_2}(q_1, q_2, q_3)}{p_2} = \frac{U_{q_3}(q_1, q_2, q_3)}{p_3}. \tag{12}$$

The partial derivative U_{q_1} is the *marginal utility* of type 1 goods. I.e., it is the change in utility due to the consumption of one additional unit of type 1 goods, assuming that the consumption of the other goods does not change.

Dividing U_{q_1} by the price p_1 gives the change in utility due to consumption of an additional dollar's worth of type 1 goods. For example, if $p_1 = 3$, then \$1 buys $\Delta q_1 = 1/3$ of a unit of type 1 goods, and the approximation formula tells us that

$$\Delta U \approx U_{q_1} \cdot \Delta q_1 = \frac{U_{q_1}}{3} = \frac{U_{q_1}}{p_1}.$$

For this reason, the ratio U_{q_1}/p_1 is (sometimes) referred to as *the marginal utility of a dollar's worth of type 1 goods.*

The same observations are true for the ratios U_{q_2}/p_2 and U_{q_3}/p_3 as well, and the triple equation (12) can be interpreted as an *equilibrium* condition:

> *At the point (q_1, q_2, q_3) of maximum utility, the marginal utilities of a dollar's worth of each good are all equal to each other.*

Equation (12) shows that at the point of maximum utility, the critical value of the multiplier λ must be equal to this common value. In the context of utility maximization the critical value of the Lagrange multiplier is (sometimes) called the *marginal utility of money*, because if Y_d increased by \$1, then the consumer's maximum utility will increase by about λ^*, the critical value of the multiplier, regardless of how the additional dollar is spent. ∎

4. *Interpreting the Lagrange multiplier.*

In the last example, we saw that the critical value of the multiplier λ has an interesting and useful interpretation in the context of utility maximization. In fact, the critical value of the multiplier *always* has an important interpretation. To see what it is, we rely on the *envelope theorem.*

Suppose we want to optimize the objective function $f(x, y, z)$ subject to the constraint $g(x, y, z) = \phi$. The Lagrangian, $F(x, y, z, \lambda; \phi) = f(x, y, z) - \lambda(g(x, y, z) - \phi)$, is a function of the four variables x, y, z and λ, and the *constraining* value ϕ, which is a parameter.

Recall that if $(x^*, y^*, z^*, \lambda^*)$ is a critical point of the Lagrangian, then $F^*(x^*, y^*, z^*, \lambda^*)$ is equal to $f(x^*, y^*, z^*) = f^*$, because the first order condition $F_\lambda = 0$ implies that the

factor $(g(x^*, y^*, z^*) - \phi) = 0$. Now, the envelope theorem states that

$$\frac{dF^*}{d\phi} = F_\phi(x^*, y^*, z^*, \lambda^*),$$

which means that

$$\frac{df^*}{d\phi} = F_\phi(x^*, y^*, z^*, \lambda^*),$$

where f^* is the optimal value of the objective function subject to the given constraint. But $F_\phi = \lambda$, because

$$F(x, y, z, \lambda; \phi) = f(x, y, z) - \lambda(g(x, y, z) - \phi) = f(x, y, z) - \lambda g(x, y, z) + \lambda\phi,$$

and this implies that

$$\frac{df^*}{d\phi} = \lambda^*. \tag{13}$$

In other words,

> *the critical value of the Lagrange multiplier λ^* gives the rate of change of the constrained optimum with respect to the constraint, and is therefore (approximately) equal to the change in the optimal value due to a one-unit change in the constraint.*

Example 7. In Example 6, the constraint is the disposable income Y_d, and the objective function is the utility, U. It follows from Equation (13) that

$$\lambda^* = \frac{dU^*}{dY_d}, \tag{14}$$

where U^* is the maximum utility given the current disposable income. If Y_d were to increase by 1 unit (e.g., \$1), then the consumer's maximum utility would increase by roughly λ^*. This is the same conclusion we drew at the end of Example 6, but the relation $\lambda^* = dU^*/dY_d$ provides more information.

Specifically, using linear approximation we see that

$$\Delta U^* \approx \frac{dU^*}{dY_d} \cdot \Delta Y_d = \lambda^* \cdot \Delta Y_d, \tag{15}$$

from which it follows that

$$\Delta Y_d \approx \frac{1}{\lambda^*} \cdot \Delta U^*.$$

In particular, if we want to increase our (maximum) utility by $\Delta U^* = 1$, it follows that disposable income will have to increase by approximately $1/\lambda^*$.

In other words, the reciprocal of the Lagrange multiplier is the (approximate) cost of increasing utility by 1 unit. Because of this, $1/\lambda^*$ is sometimes called the ***shadow price*** of utility in this type of problem. ∎

5. *Minimizing cost and maximizing output.*

Two of the most basic optimization problems in economics are the problems of minimizing the cost of producing a certain number of units, and maximizing output, given a fixed production budget. Both of these are constrained optimization problems.

Throughout the following examples, I'll assume that the critical points determined by the first order conditions yield the constrained optima that we are seeking. This is certainly true in the examples involving the Cobb-Douglas production functions.

Example 8. Suppose that a firm's production function is $Q = Q(K, L)$, where K and L are the quantities of the firm's capital and labor inputs, respectively, and Q is the firm's output. Further suppose that the prices per unit of capital and labor input are p_K and p_L, respectively, so that the cost to the firm of using K units of capital input and L units of labor input is given by $C = p_K K + p_L L$,. The optimization problem that we want to solve in this example is *minimizing the cost of producing q units.*

The objective function here is the cost function and the constraint is given by the equation $Q(K, L) = q$, so the Lagrangian for this problem is

$$F(K, L, \lambda) = p_K K + p_L L - \lambda \left(Q(K, L) - q \right).$$

The first order conditions for F lead to the following structural equations

$$p_K = \lambda Q_K(K, L) \quad \text{and} \quad p_L = \lambda Q_L(K, L),$$

and solving these equations for λ leads to the equation

$$\lambda \; = \; \frac{p_K}{Q_K(K, L)} = \frac{p_L}{Q_L(K, L)}. \tag{16}$$

By considering reciprocals, we see that (16) is equivalent to

$$\frac{Q_K(K, L)}{p_K} = \frac{Q_L(K, L)}{p_L}, \tag{17}$$

which can be interpreted as an equilibrium condition for cost minimization, as follows.

The partial derivatives Q_K and Q_L are the *marginal products* of capital and labor, respectively. Dividing each one by the price of the corresponding input gives the *marginal product of a dollar's worth of that input.* I.e., Q_K/p_K is the (approximate) change in output due to a \$1 increase in capital spending, and Q_L/p_L is likewise the (approximate) change in output due to a \$1 increase in spending on labor. Equation (17) says:

> If the cost of producing q units is minimized at the point (K^*, L^*), then the marginal product of a dollar of capital must be equal to the marginal product of a dollar of labor at that point.

Returning to the multiplier, the envelope theorem says in this case that

$$\lambda^* = \frac{dC^*}{dq},$$

where λ^* is the critical value of the multiplier and C^* is the minimum cost of producing q units. In other words λ^* is the (approximate) cost of producing the $(q + 1)$st unit, so λ^* is the firm's *marginal cost.* ∎

The *dual* problem to the one of minimizing the cost of producing a certain number of units is that of *maximizing output*, given a fixed budget.[†]

[†]These are 'dual' optimization problems in the sense that one is obtained from the other by switching the roles of constraint and objective functions.

Example 9. A firm's production function is given by the *Cobb-Douglas* function

$$Q = 36u^{1/2}v^{1/3}w^{1/4},$$

where u, v and w are the levels of the firm's three inputs and Q is the firm's output. The price per unit of the three inputs are $p_u = \$25.00$, $p_v = \$20.00$ and $p_w = \$10.00$, respectively.

Find the levels of the three inputs that maximize the firm's output, given that the firm's production budget is $\beta = \$78,000.00$

The objective function in this example is the output, and the constraint is given by the budget and the prices:

$$25u + 20v + 10w = 78000.$$

This means that the Lagrangian for this problem is

$$F(u, v, w, \lambda) = 36u^{1/2}v^{1/3}w^{1/4} - \lambda(25u + 20v + 10w - 78000),$$

and the structural equations are

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\begin{aligned}
18u^{-1/2}v^{1/3}w^{1/4} &= 25\lambda, \\
12u^{1/2}v^{-2/3}w^{1/4} &= 20\lambda, \\
9u^{1/2}v^{1/3}w^{-3/4} &= 10\lambda.
\end{aligned}
$$

Solving each of these equations for λ gives the triple equation

$$(\lambda =) \quad \frac{18v^{1/3}w^{1/4}}{25u^{1/2}} = \frac{3u^{1/2}w^{1/4}}{5v^{2/3}} = \frac{9u^{1/2}v^{1/3}}{10w^{3/4}}.$$

Comparing the first and second expressions, canceling the common factor $w^{1/4}$ and clearing denominators (cross-multiplying) gives

$$90v = 75u \implies \boxed{v = \frac{5u}{6}.}$$

Similarly, comparing the first and third expressions, canceling the common factor $v^{1/3}$ and clearing denominators gives

$$180w = 225u \implies \boxed{w = \frac{5u}{4}.}$$

Next, substituting the boxed expressions for v and w into the constraint equation gives

$$25u + 20\frac{5u}{6} + 10\frac{5u}{4} = 78000 \implies 650u = 936000 \implies \boxed{u^* = 1440.}$$

This means that $v^* = 1200$, $w^* = 1800$ and

$$\lambda^* = \frac{18(v^*)^{1/3}(w^*)^{1/4}}{25(u^*)^{1/2}} \approx 1.3133.$$

Thus, the critical point is $(u^*, v^*, w^*, \lambda^*) = (1440, 1200, 1800, 1.3133)$, and the firm's maximum output given their \$78,000.00 budget is

$$Q^* = 36(u^*)^{1/2}(v^*)^{1/3}(w^*)^{1/4} \approx 94557.42.$$

Question: By how much can the firm increase its output if their budget increases to $80,000.00?

Answer: By the envelope theorem, $dQ^*/d\beta = \lambda^*$, so by the approximation formula

$$\Delta Q^* \approx \lambda^* \Delta\beta = (1.3133) \cdot 2000 = 2662.6. \tag{18}$$

Note: If you repeat the optimization problem with $\beta = 80000,^{\ddagger}$ you will find that the new critical point is $(u^*, v^*, w^*, \lambda^*) \approx (1476.92, 1230.77, 1846.15, 1.3161)$, and the new maximum output is $Q^* = 97186.8$. Thus, the *actual* value of ΔQ^* is $97186.80 - 94557.42 = 2629.38$. The approximation in equation (18) is high by about 33 or 1.2%, which is not bad, considering the size of $\Delta\beta$. ∎

Example 10. Let's return to the problem of cost minimization, but specify that the production function is a Cobb-Douglas function. Specifically, we want to minimize the cost, $C = p_K K + p_L L$ of producing q units, where the production function has the form $Q = AK^\alpha L^\beta$. The point is to see how the partial elasticities α and β affect the firm's marginal cost function.

The first order conditions for optimization are the constraint

$$AK^\alpha L^\beta = q,$$

together with the structural equation (16), which for the Cobb-Douglas model is given explicitly as

$$(\lambda =) \qquad \frac{p_K}{\alpha A K^{\alpha-1} L^\beta} = \frac{p_L}{\beta A K^\alpha L^{\beta-1}},$$

which can be rewritten as

$$(\lambda =) \qquad \frac{p_K K}{\alpha A K^\alpha L^\beta} = \frac{p_L L}{\beta A K^\alpha L^\beta}, \tag{19}$$

because $K^{\alpha-1} = K^\alpha/K$ and $L^{\beta-1} = L^\beta/L$.

After canceling the common factors $AK^\alpha L^\beta$ from both denominators and solving the resulting *linear equation* for L, the structural equation (19) reduces to

$$L = \left(\frac{\beta}{\alpha} \cdot \frac{p_K}{p_L}\right) K. \tag{20}$$

This gives us our first conclusion, even before finding the critical point. Namely, *to minimize cost in this model,*§ *the ratio of labor input to capital input must be equal to the constant* $(\beta/\alpha) \cdot (p_K/p_L)$.

This leads to a second conclusion. The firm's capital and labor *expenditures*, E_K and E_L, are the amounts of money they spend on each input, and are given by

$$E_K = p_K K \quad \text{and} \quad E_L = p_L L.$$

Assuming that the firm acts to minimize their cost of production, so that the input levels satisfy (20), we find that the ratio of labor to capital expenditure is given by

$$\frac{E_L}{E_K} = \frac{p_L L}{p_K K} = \frac{p_L \left(\frac{\beta}{\alpha} \cdot \frac{p_K}{p_L}\right) K}{p_K K} = \frac{\beta}{\alpha}.$$

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‡which I recommend, for practice' sake.
§I.e., where output is a Cobb-Douglas function of the inputs and the prices of the inputs are *fixed*.

I.e., *assuming constant prices of inputs, the ratio of labor expenditure to capital expenditure in the Cobb-Douglas model is equal to the ratio of labor elasticity of output to capital elasticity of output.*[¶]

Returning to the problem of actually finding the critical input levels, we substitute the expression for L, above, into the *output* constraint $AK^\alpha L^\beta = q$, which gives

$$AK^\alpha \left(\frac{\beta}{\alpha} \cdot \frac{p_K}{p_L}\right)^\beta K^\beta = q \implies A\left(\frac{\beta}{\alpha} \cdot \frac{p_K}{p_L}\right)^\beta K^{\alpha+\beta} = q.$$

Solving the right-hand expression for K^* and then using (20) to compute L^*, we find that the cost-minimizing levels of capital and labor inputs are

$$K^* = \Lambda_K q^{1/(\alpha+\beta)} \quad \text{and} \quad L^* = \Lambda_L q^{1/(\alpha+\beta)}, \tag{21}$$

where the *constants* Λ_K and Λ_L are determined by the input elasticities, the prices of capital and labor inputs and the constant A:

$$\Lambda_K = A^{-\frac{1}{\alpha+\beta}} \cdot \left(\frac{\alpha \cdot p_L}{\beta \cdot p_K}\right)^{\frac{\beta}{\alpha+\beta}} \quad \text{and} \quad \Lambda_L = \left(\frac{\beta}{\alpha} \cdot \frac{p_K}{p_L}\right) \cdot \Lambda_K = A^{-\frac{1}{\alpha+\beta}} \cdot \left(\frac{\beta \cdot p_K}{\alpha \cdot p_L}\right)^{\frac{\alpha}{\alpha+\beta}}.$$

Finally, using (19) to find the critical value of the multiplier, we see that

$$\lambda^* = \frac{p_K K^*}{\alpha A (K^*)^\alpha (L^*)^\beta} = \frac{p_K \Lambda_K q^{1/(\alpha+\beta)}}{\alpha q} = \Lambda_\lambda q^{\frac{1}{\alpha+\beta}-1}, \tag{22}$$

where

$$\Lambda_\lambda = \frac{p_K \Lambda_K}{\alpha} = \left(\frac{(p_K/\alpha)^\alpha (p_L/\beta)^\beta}{A}\right)^{\frac{1}{\alpha+\beta}}.$$

As we saw in Example 8, λ^* is essentially the firm's marginal cost, so it follows from Equation (22) that the sum of the elasticities $(\alpha+\beta)$ determines the nature of the marginal cost function. Namely,

1. If $\alpha + \beta < 1$, then $\frac{1}{\alpha+\beta} - 1 > 0$, and λ^* is an increasing function of the output q, and therefore marginal cost is *increasing.*

2. If $\alpha + \beta > 1$, then $\frac{1}{\alpha+\beta} - 1 < 0$, and λ^* is a decreasing function of q, and therefore marginal cost is *decreasing.*

3. If $\alpha + \beta = 1$, then $\frac{1}{\alpha+\beta} - 1 = 0$, and λ^* is constant, and therefore marginal cost is *constant* and equal to Λ_λ.

Comments:

a. The conclusions above depend on the fairly restrictive assumption that the firm's cost is a linear function of the inputs, i.e., that the prices per unit of capital and labor input are *constant.*

b. The constants Λ_K, Λ_L and Λ_λ have somewhat complicated forms, but the important thing in this example is that they depend only on the prices, the elasticities and the constant A, and **not** depend on the constraint q. In other words they don't change with the target output. ∎

[¶]Recall that in the Cobb-Douglas model, α and β are the capital and labor elasticities of output, respectively.

6. *Second order conditions for constrained optimization.*

There are second derivative tests for constrained optimization problems as well. The principle behind these tests is the same as it was before,[‖] but the resulting formulas are more complicated because of the constraint(s) and the multiplier(s). To illustrate, I describe below the simplest case, namely the second derivative test for a constrained optimum in two variables (with one constraint).

Let (x^*, y^*, λ^*) be a critical point of the Lagrangian, $F(x, y, \lambda) = f(x,y) - \lambda(g(x,y) - c)$, and let the **discriminant** be defined by

$$D(x, y, \lambda) = 2g_x(x,y)g_y(x,y)(f_{xy}(x,y) - \lambda g_{xy}(x,y))$$
$$- \left[g_x(x,y)^2(f_{yy}(x,y) - \lambda g_{yy}(x,y)) + g_y(x,y)^2(f_{xx}(x,y) - \lambda g_{xx}(x,y)) \right].$$

Then:

(*) If $D(x^*, y^*, \lambda^*) > 0$, then $f(x^*, y^*)$ is a local maximum value of $z = f(x,y)$ subject to the constraint $g(x,y) = c$.

(*) If $D(x^*, y^*, \lambda^*) < 0$, then $f(x^*, y^*)$ is a local minimum value of $z = f(x,y)$ subject to the constraint $g(x,y) = c$.

Example 11.   Suppose that a family's disposable income is Y_d, and they want to maximize the utility $U(x,y) = a \ln x + b \ln y$ of consuming x units of good A and y units of good B, where a and b are positive parameters and p_A and p_B are the prices per unit of goods A and B, respectively. In other words, we need to find the maximum value of $U(x,y)$ subject to the constraint $g(x,y) = p_A x + p_B y = Y_d$.

The Lagrangian for this problem is

$$F(x, y, \lambda) = a \ln x + b \ln y - \lambda(p_A x + p_B y - Y_d),$$

and the structural equations are

$$F_x = \frac{a}{x} - \lambda p_A = 0$$
$$F_y = \frac{b}{y} - \lambda p_B = 0.$$

Solving these equations for λ gives

$$\lambda = \frac{a}{p_A x} = \frac{b}{p_B y},$$

and solving the right-hand equation for y gives

$$y = \frac{b p_A}{a p_B} x.$$

Substituting this into the constraint we find the critical x-value

$$p_A x + p_B \frac{b p_A}{a p_B} x = Y_d \implies x^* = \frac{Y_d}{p_A}\left(\frac{a}{a+b}\right),$$

[‖] I.e., considering the quadratic Taylor polynomial of the Lagrangian function centered at the critical point. However, there are added considerations due to the constraint.

from which we obtain the critical y and λ values,

$$y^* = \frac{Y_d}{p_B}\left(\frac{b}{a+b}\right) \quad \text{and} \quad \lambda^* = \frac{a+b}{Y_d},$$

as you should check!

In this example, the constraining function is linear $(g(x,y) = p_A x + p_B y)$ so the second derivatives g_{xx}, g_{yy} and g_{xy} in the definition of the discriminant above are *all zero*. Furthermore, $U_{xy} = 0$ too (check!), so the discriminant simplifies considerably here:

$$\begin{aligned}
D(x,y,\lambda) &= 2g_x(x,y)g_y(x,y)(U_{xy}(x,y) - \lambda g_{xy}(x,y)) \\
&\quad - \left[g_x(x,y)^2(U_{yy}(x,y) - \lambda g_{yy}(x,y)) + g_y(x,y)^2(U_{xx}(x,y) - \lambda g_{xx}(x,y))\right] \\
&= -\left[p_A^2\left(-\frac{b}{y^2}\right) + p_B^2\left(-\frac{a}{x^2}\right)\right] \\
&= \frac{p_A^2 b}{y^2} + \frac{p_B^2 a}{x^2}.
\end{aligned}$$

Since a and b are assumed to be positive parameters, the discriminant is positive *for all x and y*. In particular, we have

$$D(x^*, y^*, \lambda^*) > 0$$

so

$$U^* = U(x^*, y^*) = a\ln\left(\frac{Y_d}{p_A}\left(\frac{a}{a+b}\right)\right) + b\ln\left(\frac{Y_d}{p_B}\left(\frac{b}{a+b}\right)\right)$$

is a local maximum value. Furthermore, since the discriminant is *always positive* (in this example), this is the *absolute* maximum utility that the household can attain given its income constraint.

Exercises

1. A household's utility function is given by $U(x, y, z) = 8 \ln x + 5 \ln y + 7 \ln z$, where x, y and z are the quantities of goods of types A, B and C that the household consumes in a month. The average prices per unit of these goods are $p_A = \$10.00$, $p_B = \$6.00$ and $p_C = \$8.00$, respectively.

   (a) Find the quantities of each type of good that the household should consume in a month to *maximize* its utility, given that its monthly disposable income is $2400.00. What is the max utility that the household achieves?

   (b) By *approximately* how much does the household's disposable income have to increase, if they are to increase their utility by 2 units? Justify your answer.

2. A firm's production function is given by $Q = 32k^{3/4}l^{1/4}$, where Q is monthly output and k and l are the quantities of capital and labor input, respectively, used per month. The average monthly cost per unit of capital is $p_k = \$12,000$ and the average monthly cost per unit of labor is $p_l = \$3,000$.

   Find the quantities of capital and labor input that *minimize the cost* of producing 20,000 units of output. What is the minimal cost? What is the firm's *marginal cost* when cost is minimized? Justify your answer.

3. Another firm has the production function given by $Q = 20u^{0.3}v^{0.2}w^{0.5}$, where u, v and w are the annual quantities of the inputs U, V and W that the firm uses in to produce its output, and Q is the firm's annual output. The price per unit of inputs U, V and W are $P_U = 1000$, $P_V = 2500$ and $P_W = 4000$, respectively, and the firm's annual production budget is $B = \$1,000,000$.

   Find the quantities of each input that maximize the firm's output, subject to its budget constraint. By approximately how much will the firm's maximal output change, if its annual budget increases to $B_1 = \$1,010,000$? Justify your answer.

4. In Example 11, verify that the household's maximum utility is a linear function of the log of its income, i.e.,
   $$U^* = \alpha \ln(Y_d) + \beta.$$

   What are the constants α and β?